NUMERICAL SOLUTION OF FUZZY TWO-POINT BOUNDARY VALUE PROBLEMS VIA REPRODUCING KERNEL HILBERT SPACE METHOD

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DEDICATION

With my eternal love,

I dedicate this thesis,

to my parents, sisters and brothers,

specially my wife...

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First of, all my great, continuous thanks to Allah who gave me the ability and patience to handle the years I spent during my study despite of all the difficulties I have met.

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LIST OF SYMBOLS

Number	**Symbol Caption**	
1	$\mathbb{R}$	The Real Numbers
2	R_F	The Set of Fuzzy Numbers
3	R_F^+	The Set of Positive Fuzzy Numbers
4	$R(F)$	The Set of Fuzzy Quantities
5	Usc	Upper Semicontinuous
6	$H-derivative$	Hukuhara-derivative
7	FDE	Fuzzy Differential Equation
8	$FIVP$	Fuzzy Initial Value Problem
9	$FBVP$	Fuzzy Boundary Value Problem
10	$RKHS$	Reproducing Kernel Hilbert Space
11	$K_x(y)$	Reproducing Kernel Function
12	$W_2^m[a,b]$	Sobolev Space
13	$\{\psi_i\}_{i=1}^{\infty}$	Orthogonal Function System
14	$\{\overline{\psi}_i\}_{i=1}^{\infty}$	Orthonormal Function System
15	β_{ik}	Orthogonalization Coefficients
16	L_2	Lebesgue Space

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ABSTRACT

In this thesis, we interpret a two-point boundary value problem for a second order fuzzy differential equation by using a generalized differentiability concept. We investigate the problem of finding a numerical approximation of solutions for fuzzy two-point boundary value problems. Then we show that the reproducing kernel Hilbert space method for boundary value problems can be applied to solve numerically fuzzy two-point boundary value problems under generalized differentiability.

Keywords: Fuzzy two-point boundary value problem, Generalized differentiability, Generalized solution, Reproducing kernel Hilbert space method.

INTRODUCTION

The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1972); it was followed up by Dubois and Prade (1982), who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu (1983) and Goetschel and Voxman (1986). Kandel and Byatt (1978, 1980) applied the concept of fuzzy differential equations to the analysis of fuzzy dynamical problems. The fuzzy differential equation and the fuzzy initial value problem (Cauchy problem) were rigorously treated by Kaleva (1987, 1990), Seikkala (1987), He and Yi (1989), Kloeden (1991) and Menda (1988), Bede (2006), Buckley and Feuring (1999, 2003), Congxin and Shiji (1998), Ding (1997) and Jowers (2007). The numerical methods for solving fuzzy differential equations (FDEs) are introduced in Hüllermeier (1999), Abbasbandy and Allahviranloo (2002), Abbasbandy (2004), Babolian (2004), Bede (2007), Allahviranloo (2007) and Nieto, Khastan and Ivaz (2009).

The fuzzy boundary value problems are introduced in Lakshmikantham, Murty and Turner (2001), O'Regan, Lakshmikantham and Nieto (2003), Bede (2006), Chen, Fu, Xue and Wu (2008), Chen, Wu, Xue and Liu (2008) and Khastan and Nieto (2010).

Fuzzy set theory is a powerful tool for modelling uncertainty and for processing vague or subjective information in mathematical models. Also, fuzzy systems are useful to study a population models, the golden mean, particle systems, quantum optics and gravity, synchronize hyperchaotic systems, control chaotic systems, medicine, to bioinformatics and computational biology.

The reproducing kernel was used for the first time at the beginning of the 20th century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions. In 1907, he was the first who introduced, in a particular case, the kernel corresponding to a class of functions, and stated its reproducing property. But he did not develop any theory and did not give any particular name to the kernels he introduced.

In 1909, J. Mercer examined the functions which satisfy reproducing property in

the theory on integral equations developed by Hilbert and he called this functions as 'positive definite kernels'. He showed that this positive definite kernels have nice properties among all continuous kernels of integral equations.

However, for a long time these results were not investigated. Then the idea of reproducing kernels appeared in the dissertations of three Berlin mathematicians G. Szegŏ (1921), S. Bergman (1922) and S. Bochner (1922). In particular, S. bergman introduced reproducing kernels in one and several variables for the class of harmonic and analytic functions and he called them 'kernel functions'.

Later, the theory of reproducing kernels was systematized by Aronszajn around 1948.

The original idea of Zaremba to apply the kernels to the solution of boundary value problems was developed by S. Bergman and M. Schiffer.

There are also several papers and lecture notes on this subject; Minggen (1986), Burbea (1987), Hille (1972), Saitoh (1988), Dym (1989) and Ando (1987).

In this thesis, we present a new numerical method to solve a fuzzy two-point boundary value problem. The fuzzy two-point boundary value problem is replaced by four boundary value problems systems which are then solved numerically using reproducing kernel Hilbert space method. Also, we present numerical examples to illustrate our method.

In chapter one, we will consider the application of the lower and upper fuzzy representation to different calculations that appear in many areas of the fuzzy calculus, including defuzzification, fuzzy differentiation and integration, fuzzy differential equations and Zadeh's extension of functions.

In chapter two, we study the equivalence between a fuzzy two-point boundary value problem and a fuzzy integral equation written by using Green's function. In this sense we obtain a counterexample which shows that the two-point boundary value problem for fuzzy differential equations is not equivalent to the integral equation written by using Green's function under Hukuhara differentiability in the fuzzy differential equation, and

using fuzzy Auman-type integral in the integral equation. But under a new structure and certain conditions we show that a fuzzy two-point boundary value problem is equivalent to a fuzzy integral equation. Also, we prove that the existence of solutions to the two-point boundary value problem.

In chapter three, we give a new method to solve a system of second order two-point boundary value problems. Its exact solutions are represented in the form of series in the reproducing kernel space. The n-term approximation $u_n(x)$, $v_n(x)$ converge to the exact solutions $u(x)$, $v(x)$, respectively.

In chapter four, we present three problems to illustrate our method. Using Mathematica 7.0 we can find the reproducing kernel function $K(x, y)$ in a complete space $W_2^3[0, 1]$ and the approximate solutions for the fuzzy two-point boundary value problems.

Chapter One
Basic Concepts in Fuzzy Numbers

In this chapter, we present a representation of fuzzy numbers and its applications by Zadeh's extension principlies. Moreover, fundamental concepts in fuzzy theory are the support, level-sets (or level-cuts) and the core of a fuzzy set. Also, we present the arithmetic operations on fuzzy numbers which are usually approached either by the use of the extension principle or by the interval arithmetics as outlined by Dubois and Prade [45].

1.1 Fuzzy Numbers

Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \longrightarrow [0,1]$ and $\mu_A(x)$ interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$, i.e., a fuzzy set A of X is uniquely characterized by the pairs $(x, \mu_A(x))$ for each $x \in X$.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The mapping μ_A is also called the membership function of fuzzy set A.

A fuzzy set on $\mathbb{R}$ is a function from $\mathbb{R}$ to $[0,1]$. That is, if $A : \mathbb{R} \longrightarrow [0,1]$, then A is called a fuzzy set.

Now, we present some definitions and theorems to define the basic concepts in fuzzy numbers.

Definition 1.1.1 [43] A fuzzy set A of X is called fuzzy convex if X is convex set and for each x,$y \in X$ and $\lambda \in [0,1]$, we have

$$A(\lambda x + (1-\lambda)y) \geq \min(A(x), A(y)).$$

Definition 1.1.2 [43] A fuzzy set A of a non-empty set X is called upper semicontinuous function, denoted by Usc, if the set $\{x \in X \mid A(x) \geq \alpha\}$ is closed for each $\alpha \in [0, 1]$.

Definition 1.1.3 [33] A fuzzy set A on $\mathbb{R}$ is a fuzzy number if the following conditions hold:

1. A is upper semicontinuous.
2. $A(x) = 0$ outside some interval $[c, d]$.
3. There are real numbers $a, b : c \leq a \leq b \leq d$ for which

(i) $A(x)$ is monotonic increasing on $[c, a]$.

(ii) $A(x)$ is monotonic decreasing on $[b, d]$.

(iii) $A(x) = 1$, $a \leq x \leq b$.

We denote the set of fuzzy numbers by R_F.

In the next definition, we propose a representation of fuzzy numbers.

Definition 1.1.4 [43] Let A be a fuzzy number then for all $\alpha \in [0, 1]$, the $\alpha-$cut is defined as follows:

$$
\begin{aligned}
\alpha - cut(A) &= \begin{cases} \{x \mid x \in X,\ A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1] \\ \overline{\{x \mid x \in X,\ A(x) > 0\}} \text{ if } \alpha = 0 \end{cases} \\
&= [\underline{A}_\alpha, \overline{A}_\alpha],
\end{aligned}
$$

where $\underline{A}_\alpha = \min(\alpha - cut(A)) = \inf\limits_{\alpha \in [0,1]} A_\alpha$ and $\overline{A}_\alpha = \max(\alpha - cut(A)) = \sup\limits_{\alpha \in [0,1]} A_\alpha$.

Usually we denote $\alpha - cut(A)$ by A_α. Also a fuzzy number A is completely determined by a pair of functions $\underline{A}, \overline{A} : [0, 1] \longrightarrow \mathbb{R}$. These two functions define the endpoints of A_α and satisfy the following conditions:

1. $\underline{A} : \alpha \longrightarrow \underline{A}_\alpha \in \mathbb{R}$ is a bounded increasing left continuous function for all $\alpha \in (0, 1]$ and right continuous for $\alpha = 0$.

2. $\overline{A} : \alpha \longrightarrow \overline{A}_\alpha \in \mathbb{R}$ is a bounded decreasing left continuous function for all $\alpha \in (0,1]$ and right continuous for $\alpha = 0$.

3. $\underline{A}_\alpha \leq \overline{A}_\alpha$.

Lemma 1.1.5 [43] Suppose that $A : \mathbb{R} \longrightarrow [0,1]$ is a fuzzy set, then A is a fuzzy number if and only if the following conditions hold:

(i) The $\alpha-$cut: A_α is a closed bounded interval for each $\alpha \in [0,1]$.

(ii) The $1-$cut: $A_1 \neq \phi$.

Moreover the membership function A is defined by:

$$A(x) = \begin{cases} \alpha \ if \ x = \underline{A}_\alpha \ or \ x = \overline{A}_\alpha, \ \alpha \in (0,1) \\ 1 \qquad\qquad\qquad\qquad\qquad if \ x \in A_1 \\ 0 \qquad\qquad\qquad\qquad\qquad if \ x \notin A_0. \end{cases}$$

Definition 1.1.6 [33] A fuzzy number A is said to be positive (non-negative) if $\underline{A}(0) > 0$ ($\underline{A}(0) \geq 0$), and negative (non-positive) if $\overline{A}(0) < 0$ ($\overline{A}(0) \leq 0$).
We denote the set of positive and non-negative fuzzy numbers by R_F^+ and $R_F^+ \cup \chi_0$ respectively, where $\chi_0 = [\underline{0}, \overline{0}]$.

Now, we present some properties on a fuzzy number.
Let A, B be fuzzy numbers, then

1. $A \leq B$ iff $A_\alpha \leq B_\alpha$ (In other word $\underline{A}_\alpha \leq \underline{B}_\alpha$ and $\overline{A}_\alpha \leq \overline{B}_\alpha$, $\forall \alpha \in [0,1]$).

2. $A = B$ iff $A_\alpha = B_\alpha$ for each $\alpha \in [0,1]$, the equality of fuzzy sets is cutworthy property.

3. A is Usc iff A_α closed set for each $\alpha \in [0,1]$.

4. A is convex iff A_α convex for each $\alpha \in [0,1]$.

5. A is a fuzzy number iff A_α is compact convex subset of $\mathbb{R}$ for each $\alpha \in [0,1]$ and the cardinality of A_1 is equal one.

Definition 1.1.7 [43] A fuzzy number is a fuzzy quantity A that satisfies the following conditions:

(i) A is Usc function on $\mathbb{R}$.

(ii) The cardinality of $A_1 = \{x \in \mathbb{R} \mid A(x) = 1\}$ is equal one.

(iii) A is a fuzzy convex function.

(iv) $A_0 = \{x \in \mathbb{R} \mid A(x) > 0\}$ is bounded subset of $\mathbb{R}$.

The set of fuzzy sets of the real line are called fuzzy quantities and denoted by $R(F)$.

The next theorem characterizes fuzzy numbers through their $\alpha-$cut presentation.

Theorem 1.1.8 (**Goetschel-Voxman Representation**) [43] Let A be a fuzzy number and A_α is the $\alpha-$cut representation of A. Then $\underline{A}_\alpha$ and $\overline{A}_\alpha$ can be regarded as a real valued functions on $[0,1]$, which satisfy the following conditions:

(i) $\underline{A}_\alpha$ is bounded non-decreasing left continuous function on $(0,1]$.

(ii) $\overline{A}_\alpha$ is bounded non-increasing left continuous function on $(0,1]$.

(iii) $\underline{A}_\alpha$ and $\overline{A}_\alpha$ are right continuous at $\alpha = 0$.

(iv) $\underline{A}_1$ and $\overline{A}_1$ are equal.

1.2 Zadeh's Extension Principle

It is well known that Zadeh's extension principle play an important role in the fuzzy set theory. The Zadeh's extension is the way it produce a fuzzy transformation $F : R_F \longrightarrow R_F$ from a given function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Theorem 1.2.1 [33] Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then F is well-defined function from $R_F \times R_F$ to R_F with $\alpha-$cut

$$(F(A, B))_\alpha = f(A_\alpha, B_\alpha),$$

for every $A, B \in R_F$ and $\alpha \in [0, 1]$.

Hence a function F is called a fuzzy function induced by the extension principle.

Definition 1.2.2 [43] Let F be a fuzzy valued function on R_F. Then for $A \in R_F$, $F_\alpha[A] = (F[A])_\alpha$ for each $\alpha \in [0, 1]$ is called the $\alpha-$cut function of F. And so, $F_\alpha[A] = \{x \in \mathbb{R} \mid F[A](x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $F_0[A] = \{x \in \mathbb{R} \mid F[A](x) > 0\}$ if $\alpha = 0$.

The following definition gives the method to extend a function defined in Euclidean space $T \times \mathbb{R} \times \mathbb{R}$ to a fuzzy space $T \times R_F \times R_F$,where $T \subseteq \mathbb{R}$.

Definition 1.2.3 [36] The Zadeh's extension of a function $f : T \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is the function $F : T \times R_F \times R_F \longrightarrow R_F$ with $\alpha-$cut

$$F_\alpha(t, A, B) = f(t, A_\alpha, B_\alpha) = \{f(t, a, b) \mid a \in A_\alpha, b \in B_\alpha\},$$

for each $\alpha \in [0, 1]$, $t \in T$ and $A, B \in R_F$.

The above theorem and definitions in the Zadeh's extension principle can be used to define the arithmetic operations and some basic functions on the set of fuzzy numbers and it is widely used in solving fuzzy differential equations (FDEs).

1.3 Arithmetic Operations on Fuzzy Numbers

The basic arithmetic operations between two closed bounded intervals are defined by

$$A \circ B = \{a \circ b \mid a \in A, b \in B\},$$

where $\circ \in \{+, -, \times, \div\}$ and the division case we require that $0 \notin B$.

For $\circ \in \{+, -, \times, \div\}$ and A, B are closed bounded intervals of $\mathbb{R}$, then $A \circ B$ is a closed bounded interval.

The arithmetic operations for two fuzzy numbers $A = [\underline{A}, \overline{A}]$ and $B = [\underline{B}, \overline{B}]$ are defined in the standard way, in terms of the $\alpha-$cuts for $\alpha \in [0, 1]$:

Addition: $(A + B)_\alpha = [\underline{A} + \underline{B}, \overline{A} + \overline{B}]$;

Scalar Multiplication: for given $\lambda \in \mathbb{R}$,

$$(\lambda A)_\alpha = \begin{cases} [\lambda\underline{A}, \lambda\overline{A}] \text{ if } \lambda \geq 0 \\ [\lambda\overline{A}, \lambda\underline{A}] \text{ if } \lambda < 0; \end{cases}$$

Subtraction: $(A - B)_\alpha = [\underline{A} - \overline{B}, \overline{A} - \underline{B}]$;

Multiplication: $(AB)_\alpha = [\min X, \max X]$, where $X = \{\underline{A}\ \underline{B}, \underline{A}\ \overline{B}, \overline{A}\ \underline{B}, \overline{A}\ \overline{B}]$;

Division: $(A \div B)_\alpha = A \times \dfrac{1}{B} = [\min X, \max X]$, where $X = \{\dfrac{\underline{A}}{\underline{B}}, \dfrac{\underline{A}}{\overline{B}}, \dfrac{\overline{A}}{\underline{B}}, \dfrac{\overline{A}}{\overline{B}}\}$, provided that $0 \notin B$.

1.4 Some Applications on Fuzzy Numbers Using Zadeh's Extension Principle

First application 1.4.1 [47] The absolute value of a fuzzy number $X \in R_F$ is a function $F : R_F \longrightarrow R_F$ denoted by $F(X) = |X|$ with $\alpha-$cut $(|X|)_\alpha = \{|x| \mid x \in X_\alpha\}$. From the interval analysis, we know that if $I = [\underline{I}, \overline{I}]$, then

$$|I| = [\max(\underline{I}, -\overline{I}, 0), \max(-\underline{I}, \overline{I})],$$

thus the $\alpha-$cut of $|A|$ is given by

$$(|A|)_\alpha = [\max(\underline{A}_\alpha, -\overline{A}_\alpha, 0), \max(-\underline{A}_\alpha, \overline{A}_\alpha)].$$

Since $f(x) = |x|$ is a continuous function on $\mathbb{R}$, we get $F(X) = |X|$ is a continuous function on R_F.

Second application 1.4.2 [47] The square root of a fuzzy number $X \in R_F$ is a function

$F : R_{U_F} \longrightarrow R_F$ denoted by $F(X) = \sqrt{X}$ with $\alpha-$cut $(\sqrt{X})_\alpha = \{\sqrt{x} \mid x \in X_\alpha\}$, where $U = [0, \infty)$ and $R_{U_F} = \{X \in R_F \mid X_0 \subseteq U\}$.

Since $f(x) = \sqrt{x}$ is a continuous function on U, we get $F(X) = \sqrt{X}$ is a continuous function on R_{U_F}. Because f is increasing on U, we get

$$(\sqrt{X})_\alpha = [\, \underline{\sqrt{X}}_\alpha, \overline{\sqrt{X}}_\alpha].$$

Theorem 1.4.3 [47] Let $A \in R_F$, then we have

(i) $\sqrt[n]{A^n} = |A|$ if n is an even positive integer.

(ii) $\sqrt[n]{A^n} = A$ if n is an odd positive integer.

Note that $\sqrt{A^2}$ is not necessarily equal to A.

Now, we give some properties of the square root of a fuzzy number. Observe that these properties are similar to the square root of the real numbers.

Theorem 1.4.4 [47] Let $A, B \in R_F^+ \cup \chi_0$, then we have

(i) $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$.

(ii) $\sqrt{AB} = \sqrt{A}\sqrt{B}$.

(iii) $\sqrt{\dfrac{A}{B}} = \dfrac{\sqrt{A}}{\sqrt{B}}$, provided that $0 \notin B_0$.

Third application 1.4.5 [47] The exponential of a fuzzy number $X \in R_F$ is a function $F : R_F \longrightarrow R_F$ denoted by $F(X) = \exp(X)$ with $\alpha-$cut $(\exp(X))_\alpha = \{\exp(x) \mid x \in X_\alpha\}$.

Since $f(x) = e^x$ is a continuous function on $\mathbb{R}$, we get $F(X) = \exp(X)$ is a continuous function on R_F. Because f is increasing on $\mathbb{R}$, we get

$$(\exp(X))_\alpha = [\, \underline{\exp(X)}_\alpha, \overline{\exp(X)}_\alpha].$$

Fourth application 1.4.6 [47] The natural logarithm of a fuzzy number $X \in R_{\mathcal{F}}$ is a function $F : R_{\mathcal{F}}^{+} \longrightarrow R_{\mathcal{F}}$ denoted by $F(X) = \ln X$ with $\alpha-$cut $(\ln X)_\alpha = \{\ln x \mid x \in X_\alpha\}$.

Since $f(x) = \ln x$ is a continuous function on $[\epsilon, \infty)$, $\epsilon > 0$, we get $F(X) = \ln X$ is a continuous function on $R_{\mathcal{F}}^{+}$. Because f is increasing on $[\epsilon, \infty)$, we get

$$(\ln X)_\alpha = [\, \underline{\ln X}_\alpha, \overline{\ln X}_\alpha].$$

Next, we present some properties for the natural logarithm of a fuzzy number. These properties are similar to the natural logarithm of the real numbers.

Theorem 1.4.7 [47] For any positive fuzzy numbers A and B and a rational number r, we have

(i) $\ln AB = \ln A + \ln B$.

(ii) $\ln \dfrac{A}{B} = \ln A - \ln B$.

(iii) $\ln A^r = r \ln A$.

Note that the theorem that we stated in the previous is also hold for the logarithm to any base of a fuzzy number. In this case $F : R_{\mathcal{F}}^{+} \longrightarrow R_{\mathcal{F}}$ denoted by $F(X) = \log_b X$ with $\alpha-$cut $(\log_b X)_\alpha = \{\log_b x \mid x \in X_\alpha\}$ and $b \in \mathbb{R}^+$.

1.5 Levelwise Continuity

For A, $B \in R_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $A + B$ and the product λA are defined by $[A + B]_\alpha = [A]_\alpha + [B]_\alpha$, $[\lambda A\ ]_\alpha = \lambda [A]_\alpha$, $\forall \alpha \in [0, 1]$, where $[A]_\alpha + [B]_\alpha$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda [A]_\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$. The metric structure is given by the Hausdorff distance $d : R_{\mathcal{F}} \times R_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$, $d(u, v) = \sup\limits_{\alpha \in [0,1]} \max\{|\underline{A}_\alpha - \underline{B}_\alpha|, |\overline{A}_\alpha - \overline{B}_\alpha|\}$, and it is well known that the $(R_{\mathcal{F}}, d)$ is a complete space.

A fuzzy valued mapping $F : T \longrightarrow R_F$ is continuous at $t_0 \in T$ if for every $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that: $d(F(t), F(t_0)) < \epsilon$, for all $t \in T$ with $\|t - t_0\| < \delta$.

Theorem 1.5.1 [43] Let $F : T \longrightarrow R_F$, where $[F(t)]_\alpha = [\underline{F(t)}_\alpha, \overline{F(t)}_\alpha]$, then F is continuous on T if and only if both $\underline{F(t)}_\alpha$ and $\overline{F(t)}_\alpha$ are continuous on T.

Theorem 1.5.2 [33] Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Then the following conditions are equivalent:

(i) f is continuous.

(ii) $F : R_F \times R_F \longrightarrow R_F$ is continuous with respect to the metric d.

Definition 1.5.3 [43] Let F be a fuzzy valued function on a compact convex subset T of $\mathbb{R}$. Suppose that the $\alpha-$cut function F_α is continuous at $t_0 \in T$ for each $\alpha \in [0, 1]$. Then we say that F is levelwise continuous at $t = t_0$.

If F is levelwise continuous at every point of T, then we call F is levelwise continuous on T.

The relationship between the levelwise continuity of the fuzzy valued function F and the continuity of its $\alpha-$cut functions $\underline{F(t)}_\alpha$ and $\overline{F(t)}_\alpha$ for each $\alpha \in [0, 1]$ will be give in the next theorem.

Theorem 1.5.4 [43] Let F be a fuzzy valued function on a compact convex T. Then F is levelwise continuous on T if and only if the real valued functions $\underline{F(t)}_\alpha$ and $\overline{F(t)}_\alpha$ are continuous on T for each $\alpha \in [0, 1]$.

1.6 Fuzzy Differentiation

It well known that the H $-$derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu [46] and it is based in the $H-$difference of sets, as follows:

For $u,\, v \in R_{\mathcal{F}}$, $w \in R_{\mathcal{F}}$ is called the Hukuhara difference of u and v if $u = v + w$, and it is denoted by $w = u - v$.

Definition 1.6.1 [32] Let $F : T \longrightarrow R_{\mathcal{F}}$ and $t_0 \in T \subseteq \mathbb{R}$. F is differentiable at t_0, if:

1. there exist an element $F'(t_0) \in R_{\mathcal{F}}$ such that for all $h > 0$ sufficiently near to zero, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits

$$\lim_{h \longrightarrow 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0). \tag{1.1}$$

or

2. there exist an element $F'(t_0) \in R_{\mathcal{F}}$ such that for all $h < 0$ sufficiently near to zero, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits

$$\lim_{h \longrightarrow 0^-} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \longrightarrow 0^-} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0). \tag{1.2}$$

Here the limit is taken in the metric space $(R_{\mathcal{F}}, d)$.

The above definition is a straightforward generalization of the Hukuhara differentiability of a set-valued function. So if F is differentiable at $t_0 \in T$, then all its $\alpha-$levels $F_\alpha(t) = [F(t)]_\alpha$ are Hukuhara differentiable at t_0 and $[F'(t_0)]_\alpha = DF_\alpha(t_0)$, where DF_α denotes the Hukuhara derivative of F_α.

Next we shall write some properties for the second form (1.2) whose demonstrations are similar to the previous ones.

Theorem 1.6.2 [30] If F is differentiable in the second form (1.2), then it is continuous.

Theorem 1.6.3 [30] If F, G are differentiable in the second form (1.2) at the point t and $\lambda \in \mathbb{R}$, then $(F+G)'(t) = F'(t) + G'(t)$ and $(\lambda F)'(t) = \lambda F'(t)$.

In the special case when F is a fuzzy number valued mapping, we have the following theorem.

Theorem 1.6.4 [30] Let $F : T \subseteq \mathbb{R} \longrightarrow R_F$ be a function and denote $[F(t)]_\alpha = [f_\alpha(t), g_\alpha(t)]$ for each $\alpha \in [0,1]$. Then

(i) If F is differentiable in the first form (1.1), then f_α and g_α are differentiable functions, $[F'(t)]_\alpha = [f'_\alpha(t), g'_\alpha(t)]$ and $f'_\alpha(t) \leq g'_\alpha(t)$.

(ii) If F is differentiable in the second form (1.2), then f_α and g_α are differentiable functions, $[F'(t)]_\alpha = [g'_\alpha(t), f'_\alpha(t)]$ and $g'_\alpha(t) \leq f'_\alpha(t)$.

Proof. If $h < 0$ and $\alpha \in [0,1]$, then we have

$$\begin{aligned} \frac{[F(t+h) - F(t)]_\alpha}{h} &= \frac{1}{h}\left[f_\alpha(t+h) - f_\alpha(t), g_\alpha(t+h) - g_\alpha(t)\right] \\ &= \left[\frac{g_\alpha(t+h) - g_\alpha(t)}{h}, \frac{f_\alpha(t+h) - f_\alpha(t)}{h}\right]. \end{aligned}$$

Similarly we obtain

$$\frac{[F(t) - F(t-h)]_\alpha}{h} = \left[\frac{g_\alpha(t) - g_\alpha(t-h)}{h}, \frac{f_\alpha(t) - f_\alpha(t-h)}{h}\right].$$

Passing to the limit we have

$$[F'(t)]_\alpha = [g'_\alpha(t), f'_\alpha(t)],$$

and the proof is now complete.

Definition 1.6.5 [43] Let F be a fuzzy valued function on a compact convex subset T of $\mathbb{R}$. Suppose that the $\alpha-$cut function F_α is differentiable at $t_0 \in T$ for each $\alpha \in [0,1]$. If F'_α define a fuzzy number, then we say that F is levelwise differentiable at $t = t_0$.

If F is levelwise differentiable at any point $t \in T$, we call F is levelwise differentiable on T.

Theorem 1.6.6 [43] Let F be a levelwise differentiable fuzzy valued function on a compact convex subset T of $\mathbb{R}$. Then $\underline{F}_\alpha$ and $\overline{F}_\alpha$ are differentiable functions on T for each $\alpha \in [0,1]$ and $F'_\alpha(t) = [F'(t)]_\alpha = [\underline{F'(t)}_\alpha, \overline{F'(t)}_\alpha]$.

Theorem 1.6.7 [43] Let F be a levelwise differentiable fuzzy valued function on a compact convex subset T of $\mathbb{R}$, then it is levelwise continuous function on T.

1.7 Fuzzy Integration

Integrals of fuzzy set valued functions as a natural generalization of set-valued functions have been established by Kaleva [31], Puri and Ralescu [46].

In this section, we propose some propositions and theorems to prove the existence of solutions to fuzzy two-point boundary value problem under certain conditions.

We know that $C[0,1] \times C[0,1]$ is a Banach space and in $C[0,1] \times C[0,1]$, define a norm $\|z\| = \max\limits_{0 \leq \alpha \leq 1} \max\{|x(\alpha)|, |y(\alpha)|\}$, $\forall z = (x,y) \in C[0,1] \times C[0,1]$.We denote partition of $C[0,1] \times C[0,1]$ by X.

Theorem 1.7.1 [2] R_F is a closed convex cone in Banach space X, and then it is a complete metric space.

Let $F : T \longrightarrow R_F$ or X, $J \in X$. If for any partition $\triangle : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ of $T = [a,b]$, and $\forall \tau_k \in [t_{k-1}, t_k]$ $(k = 1, 2, ..., n)$, we have $\lim\limits_{\lambda(\triangle) \longrightarrow 0} \sum\limits_{k=1}^{n} F(\tau_k) \triangle t_k = J$, where $\lambda(\triangle) = \max\limits_{1 \leq k \leq n} \{\triangle t_k\}$, $\triangle t_k = t_k - t_{k-1}$, $(k = 1, 2, ..., n)$. Then F is said to be integrable on T, and denote $\int\limits_a^b F(t)dt = J$.

Proposition 1.7.2 [37] Let $F : T \longrightarrow R_F$ be integrable on $T = [a,b]$, then

$$\left[\int_a^b F(t)dt\right]_\alpha = \left[\int_a^b \underline{F(t)}_\alpha dt, \int_a^b \overline{F(t)}_\alpha dt\right], \ \alpha \in [0,1],$$

and $\int\limits_a^b F(t)dt \in R_F$.

Proposition 1.7.3 [37] Let $F : T \longrightarrow R_{\mathcal{F}}$ be integrable on $T = [a, b]$, $a < c < b$, then F is integrable both on $[a, c]$ and $[c, b]$, and $\int_a^b F(t)dt = \int_a^c F(t)dt + \int_c^b F(t)dt$.

Proposition 1.7.4 [37] Let $F : T \longrightarrow R_{\mathcal{F}}$ be continuous on $T = [a, b]$, then F is integrable on T.

Proposition 1.7.5 [2] Let $F, G : T \longrightarrow R_{\mathcal{F}}$ be integrable on $T = [a, b]$, then

$$\int_a^b [c_1F(t) + c_2G(t)]\, dt = c_1 \int_a^b F(t)dt + c_2 \int_a^b G(t)dt,\ \ c_1, c_2 \in \mathbb{R}.$$

Proposition 1.7.6 [2] Let $F : T \longrightarrow R_{\mathcal{F}}$ be continuous on $T = [a, b]$, then $\left\| \int_a^b F(t)dt \right\|$ $\leq \int_a^b \|F(t)\|\, dt$.

Theorem 1.7.7 [2] Let $F : T \longrightarrow R_{\mathcal{F}}$ be continuous on $T = [a, b]$, then $\frac{d}{dt}\left(\int_a^t F(\tau)d\tau \right) =$ $F(t)$, $t \in T$.

Corollary 1.7.8 (**Newton-Leibniz formula**) [2] Assume that $F : T \longrightarrow R_{\mathcal{F}}$ is continuously derivable on T, then $\int_a^b F'(t)dt = F(b) - F(a)$, where $F(b) - F(a)$ is also the $H-$difference of $F(b)$ and $F(a)$.

Theorem 1.7.9 [2] Let $g : T \longrightarrow \mathbb{R}$ be continuously derivable and $f : T \longrightarrow R_{\mathcal{F}}$ be continuously derivable, then

$$\int_a^b f(t)g'(t)dt = [g(t)f(t)]_a^b - \int_a^b g(t)f'(t)dt.$$

Theorem 1.7.10 [2] Assume that $g : T \longrightarrow \mathbb{R}$ is derivable and $f : T \longrightarrow R_{\mathcal{F}}$ is derivable, then $gf : T \longrightarrow R_{\mathcal{F}}$ is derivable and

$$(g(t)f(t))' = g'(t)f(t) + g(t)f'(t), t \in T.$$

Definition 1.7.11 [43] Let F be a fuzzy valued function on a compact convex subset T of $\mathbb{R}$. Suppose that the $\alpha-$cut function F_α is integrable on T for each $\alpha \in [0,1]$. Let $I_\alpha = \int\limits_T F_\alpha(t)dt$. Then we say that F is levelwise integrable on T.

Theorem 1.7.12 [43] $F : T \longrightarrow R_F$ is a levelwise integrable fuzzy valued function on a compact convex subset T of $\mathbb{R}$ if and only if $\underline{F}_\alpha$ and $\overline{F}_\alpha$ are integrable functions on T for each $\alpha \in [0,1]$.

Theorem 1.7.13 [43] Let F be a levelwise continuous fuzzy valued function on a compact convex subset T of $\mathbb{R}$, then it is levelwise integrable function on T.

1.8 Fuzzy Differential Equations

We study the Cauchy problem for differential equations, considering its parameters and/ or initial conditions given by fuzzy sets, using Hukuhara derivative. We will see that both derivatives are different and they lead us to different solutions from a fuzzy differential equations.

Let us consider the fuzzy initial value problem (FIVP)

$$\left.\begin{aligned} y'(t) &= F(t, y(t)), \\ y(t_0) &= y_0, \end{aligned}\right\} \tag{1.3}$$

where $F : [t_0, t_0 + a] \times R_F \longrightarrow R_F$ and $y_0 \in R_F$.

Let $[y(t)]_\alpha = \left[\underline{y(t)}_\alpha, \overline{y(t)}_\alpha\right]$. If we consider $y'(t)$ by using the derivative in the first form (1.1), then we have $[y'(t)]_\alpha = \left[\underline{y'(t)}_\alpha, \overline{y'(t)}_\alpha\right]$. So (1.3) translates into the following system of ordinary differential equations.

$$\left.\begin{aligned} \underline{y'(t)}_\alpha &= \underline{F}_\alpha(t, \underline{y(t)}_\alpha, \overline{y(t)}_\alpha), \\ \overline{y'(t)}_\alpha &= \overline{F}_\alpha(t, \underline{y(t)}_\alpha, \overline{y(t)}_\alpha), \\ \underline{y(t_0)}_\alpha &= \underline{y_0}_\alpha, \\ \overline{y(t_0)}_\alpha &= \overline{y_0}_\alpha, \end{aligned}\right\} \tag{1.4}$$

where $[F(t, y(t))]_\alpha = \left[\underline{F}_\alpha(t, \underline{y(t)}_\alpha, \overline{y(t)}_\alpha), \overline{F}_\alpha(t, \underline{y(t)}_\alpha, \overline{y(t)}_\alpha)\right]$.

In the following theorem we show that the FIVP (1.3) will be equivalent to the system (1.4).

Theorem 1.8.1 [8] Let us consider the FIVP (1.3) where $F : [t_0, t_0 + a] \times R_{\mathcal{F}} \longrightarrow R_{\mathcal{F}}$ is such that:

(i) $[F(t, y(t))]_\alpha = \left[\underline{F}_\alpha(t, \underline{y(t)}_\alpha, \overline{y(t)}_\alpha), \overline{F}_\alpha(t, \underline{y(t)}_\alpha, \overline{y(t)}_\alpha)\right]$.

(ii) $\underline{F}_\alpha$ and $\overline{F}_\alpha$ are equicontinuous (i.e. for any $\epsilon > 0$ and any $(t, x, y) \in [t_0, t_0+a] \times \mathbb{R}^2$ we have $|\underline{F}_\alpha(t, x, y) - \underline{F}_\alpha(t_1, x_1, y_1)| < \epsilon$ and $\left|\overline{F}_\alpha(t, x, y) - \overline{F}_\alpha(t_1, x_1, y_1)\right| < \epsilon$, $\forall \alpha \in [0, 1]$, whenever $\|(t, x, y) - (t_1, x_1, y_1)\| < \delta$) and uniformly bounded on any bounded set.

(iii) there exist $L > 0$ such that $|\underline{F}_\alpha(t, x_1, y_1) - \underline{F}_\alpha(t, x_2, y_2)| \leq L \max\{|x - u|, |y - v|\}$ and $\left|\overline{F}_\alpha(t, x_1, y_1) - \overline{F}_\alpha(t, x_2, y_2)\right| \leq L \max\{|x - u|, |y - v|\}$, $\forall \alpha \in [0, 1]$.

Then the FIVP (1.3) and the system of ordinary differential equations (1.4) are equivalent.

It is clear that in this procedure the uniqueness of the solution is lost, but it is an expected situation in the fuzzy context. Nevertheless, we can speak of the existence and uniqueness of two solutions, as is showed in the following result.

Theorem 1.8.2 [30] Let $F : T \times R_{\mathcal{F}} \longrightarrow R_{\mathcal{F}}$ be continuous and assume that there exists a $k > 0$ such that $d(F(t, u), F(t, v)) \leq kd(u, v)$, $\forall t \in T, u, v \in R_{\mathcal{F}}$. Then the problem

$$\begin{aligned} y'(t) &= F(t, y(t)), \\ y(t_0) &= y_0, \end{aligned}$$

has two unique solutions on T.

Chapter Two
Fuzzy Two-Point Boundary Value Problems

The statement that a two-point boundary value problem of fuzzy differential equation is equivalent to a fuzzy integral equation was pointed out by Lakshmikantham and O'Regan [4]. Recently, Bede [3] gave a counterexample to show that this statement does not hold and he also argued that in many cases two-point boundary value problems have no solutions. Under a new structure and certain conditions we show that a two-point boundary value problem is equivalent to a fuzzy integral equation.

2.1 Concept of Solution for the Fuzzy Two-Point Boundary Value Problem

In this section we study the fuzzy boundary value problem using the concept of differentiability. So, we recall some definitions and important theorem for a second order derivative based on the selection of derivative type in each step of differentiation. Also, we present a concept of solution for the fuzzy two-point boundary value problem. Consider the two-point boundary value problem

$$y''(t) = f(t, y(t), y'(t)), \tag{2.1}$$

with boundary conditions:

$$y(0) = \gamma_0, y(1) = \gamma_1, \tag{2.2}$$

where $\gamma_0, \gamma_1 \in R_F$ and $f : [0,1] \times R_F \times R_F \longrightarrow R_F$ is a continuous fuzzy function.

Definition 2.1.1 [1] Let $F : (a,b) \longrightarrow R_F$ and $n, m \in \{1,2\}$. We say F is $(n,m)-$differentiable at $t_0 \in (a,b)$, if $D_n^1 F$ exist on a neighborhood of t_0 as a fuzzy function and it is $(m)-$differentiable at t_0. The second derivatives of F is denoted by $D_{n,m}^2 F(t_0)$ for $n, m \in \{1,2\}$.

Theorem 2.1.2 [1] Let $D_1^1F : (a,b) \longrightarrow R_F$ or $D_2^1F : (a,b) \longrightarrow R_F$ be fuzzy functions, where $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$. Then

(i) If D_1^1F is $(1)-$differentiable, then f'_α and g'_α are differentiable functions and $[D_{1,1}^2F(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.

(ii) If D_1^1F is $(2)-$differentiable, then f'_α and g'_α are differentiable functions and $[D_{1,2}^2F(t)]^\alpha = [g''_\alpha(t), f''_\alpha(t)]$.

(iii) If D_2^1F is $(1)-$differentiable, then f'_α and g'_α are differentiable functions and $[D_{2,1}^2F(t)]^\alpha = [g''_\alpha(t), f''_\alpha(t)]$.

(iv) If D_2^1F is $(2)-$differentiable, then f'_α and g'_α are differentiable functions and $[D_{2,2}^2F(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.

Definition 2.1.3 [1] Let $y : [0,1] \longrightarrow R_F$ be a fuzzy function and $n, m \in \{1,2\}$. We say y is a $(n,m)-$solution for problem $(2.1)-(2.2)$ on $[0,1]$, if D_n^1y, $D_{n,m}^2y$ exist on $[0,1]$, $D_{n,m}^2y(t) = f(t, y(t), D_n^1y(t))$ and $y(0) = \gamma_0$, $y(1) = \gamma_1$.

Definition 2.1.4 [1] Let $y : [0,1] \longrightarrow R_F$ be a fuzzy function and $n, m \in \{1,2\}$. We say y is a $(n,m)-$solution for (2.1) on an interval $I \subset [0,1]$, if D_n^1y, $D_{n,m}^2y$ exist and $D_{n,m}^2y(t) = f(t, y(t), D_n^1y(t))$ on I.

Definition 2.1.5 [1] Let $n, m, n^*, m^* \in \{1,2\}$. Suppose $y : [0,1] \longrightarrow R_F$ and $t_0 \in (0,1)$ are such that $y(0) = \gamma_0$, y is a $(n,m)-$solution of Eq.(2.1) on $(0,t_0)$, y is a $(n^*,m^*)-$solution of Eq.(2.1) on $(t_0,1)$ and $y(1) = \gamma_1$. Then we say that y is a generalized solution of the boundary value problem $(2.1)-(2.2)$.

Let y be a $(n,m)-$solution for the problem $(2.1)-(2.2)$. By theorems 1.6.4 and 2.1.2, we can translate problem $(2.1)-(2.2)$ to a system of boundary value problems hereafter, called corresponding $(n,m)-$system for problem $(2.1)-(2.2)$.
Therefore, four BVPs systems are possible for problem $(2.1)-(2.2)$, as follow:

$(1,1)-$system:

$$\underline{y}''_\alpha(t) = \underline{f}(t, y_\alpha(t), D_1^1 y_\alpha(t)),$$
$$\overline{y}''_\alpha(t) = \overline{f}(t, y_\alpha(t), D_1^1 y_\alpha(t)),$$
$$\underline{y}_\alpha(0) = \underline{\gamma}_{0\alpha}, \overline{y}_\alpha(0) = \overline{\gamma}_{0\alpha},$$
$$\underline{y}_\alpha(1) = \underline{\gamma}_{1\alpha}, \overline{y}_\alpha(1) = \overline{\gamma}_{1\alpha},$$

$(1,2)-$system:

$$\overline{y}''_\alpha(t) = \underline{f}(t, y_\alpha(t), D_1^1 y_\alpha(t)),$$
$$\underline{y}''_\alpha(t) = \overline{f}(t, y_\alpha(t), D_1^1 y_\alpha(t)),$$
$$\underline{y}_\alpha(0) = \underline{\gamma}_{0\alpha}, \overline{y}_\alpha(0) = \overline{\gamma}_{0\alpha},$$
$$\underline{y}_\alpha(1) = \underline{\gamma}_{1\alpha}, \overline{y}_\alpha(1) = \overline{\gamma}_{1\alpha},$$

$(2,1)-$system:

$$\overline{y}''_\alpha(t) = \underline{f}(t, y_\alpha(t), D_2^1 y_\alpha(t)),$$
$$\underline{y}''_\alpha(t) = \overline{f}(t, y_\alpha(t), D_2^1 y_\alpha(t)),$$
$$\underline{y}_\alpha(0) = \underline{\gamma}_{0\alpha}, \overline{y}_\alpha(0) = \overline{\gamma}_{0\alpha},$$
$$\underline{y}_\alpha(1) = \underline{\gamma}_{1\alpha}, \overline{y}_\alpha(1) = \overline{\gamma}_{1\alpha},$$

$(2,2)-$system:

$$\underline{y}''_\alpha(t) = \underline{f}(t, y_\alpha(t), D_2^1 y_\alpha(t)),$$
$$\overline{y}''_\alpha(t) = \overline{f}(t, y_\alpha(t), D_2^1 y_\alpha(t)),$$
$$\underline{y}_\alpha(0) = \underline{\gamma}_{0\alpha}, \overline{y}_\alpha(0) = \overline{\gamma}_{0\alpha},$$
$$\underline{y}_\alpha(1) = \underline{\gamma}_{1\alpha}, \overline{y}_\alpha(1) = \overline{\gamma}_{1\alpha}.$$

Our strategy of solving $(2.1)-(2.2)$ is based on the selection of derivative type in the fuzzy boundary value problem. We choose the type of solution and translate problem $(2.1)-(2.2)$ to the corresponding system of boundary value problems and we find such a domain in which the solution and its derivatives have valid level sets according to the type of differentiability.

2.2 Existence and Uniqueness of Fuzzy Two-Point Boundary Value Problem

The boundary value problems of fuzzy differential equations can be discussed by means of the theory of abstract functions in Banach spaces. An advantage of doing in this way is that we can use usual differential methods and integration by parts. Under this new structure and certain conditions, we prove that a two-point boundary value problem of fuzzy differential equation is equivalent to a fuzzy integral equation and further get some results about existence and uniqueness of solutions.

We discuss the two-point boundary value problem

$$\left.\begin{aligned} x''(t) &= f(t,x,x'), \\ x(a) = A,&\ \ x(b) = B, \end{aligned}\right\} \tag{2.3}$$

where t belongs to any partition $\triangle : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ of $T = [a,b]$, $A, B \in R_F$, $f \in C(T \times R_F \times R_F, R_F)$.

Now, we will prove the existence of solutions to two-point boundary value problem (2.3) under certain conditions.

Definition 2.2.1 [2] If $x : T \longrightarrow R_F$ has continuous second derivative and satisfies (2.3), then $x(t)$ is said to be a solution of (2.3).

Since a solution $x(t)$ of problem (2.3) satisfies $x(t) = A + \int\limits_a^t x'(\tau)d\tau$ $(a \leq t \leq b)$, we have $B - A \in R_F$, i.e., the $H-$difference of A and B exists. Hence without loss of generality, we only consider problem (2.4) below because the solution of problem (2.3) can be derived from

$$\left.\begin{aligned} x''(t) &= f(t,x,x'), \\ x(a) = 0,&\, x(b) = B, \end{aligned}\right\} \tag{2.4}$$

where t belongs to any partition $\triangle : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ of $T = [a, b]$, $B \in R_{\mathcal{F}}$, $f \in C(T \times R_{\mathcal{F}} \times R_{\mathcal{F}}, R_{\mathcal{F}})$.

We consider the following fuzzy integral equation:

$$x(t) + \int_a^b G(t,s) f(s, x(s), x'(s)) ds = w(t), \tag{2.5}$$

where $G(t,s)$ is the Green's function:

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{b-a}, a \leq s \leq t \leq b, \\ \dfrac{(b-s)(t-a)}{b-a}, a \leq t \leq s \leq b, \end{cases}$$

and $w(t) = \dfrac{B(t-a)}{b-a}$. Then it is obvious that $w''(t) = 0$, $w(a) = 0$, $w(b) = B$,

$$\int_a^b |G(t,s)| \, ds \leq \frac{(b-a)^2}{8}, \quad \int_a^b \left| \frac{\partial G(t,s)}{\partial t} \right| ds \leq \frac{b-a}{2}.$$

Definition 2.2.2 [2] If $x : T \longrightarrow R_{\mathcal{F}}$ has continuous second derivative and satisfies (2.4) (or (2.3)), then $x(t)$ is said to be a solution of (2.4) (or (2.3)). If $x(t)$ has continuous derivative and satisfies (2.5), then $x(t)$ is said to be a solution of (2.5).

Theorem 2.2.3 [2] Let $f \in C(T \times R_{\mathcal{F}} \times R_{\mathcal{F}}, R_{\mathcal{F}})$, $f = [f_1, f_2]$, and f_1, f_2 are absolutely continuous on $[0, 1]$ with respect to α and there exists $M : [0, 1] \longrightarrow \mathbb{R}^+$ such that $\forall (t, u, v) \in T \times R_{\mathcal{F}} \times R_{\mathcal{F}}$,

$$\left| \frac{\partial}{\partial \alpha} \underline{f}(t, u, v, \alpha) \right|, \left| \frac{\partial}{\partial \alpha} \overline{f}(t, u, v, \alpha) \right| \leq M(\alpha), \text{ a.e. } \alpha \in [0, 1].$$

Let $B = [B_1, B_2] \in R_{\mathcal{F}}$ where B_1 and B_2 are absolutely continuous on $[0, 1]$, and

$$\left| \frac{dB_1}{d\alpha} \right|, \left| \frac{dB_2}{d\alpha} \right| \geq (b-a)^2 M(\alpha), \text{ a.e. } \alpha \in [0, 1],$$

then $x(t)$ is a solution of (2.4) if and only if $x(t)$ is a solution of (2.5).

Proof. Let $x(t)$ be a solution of (2.4), then

$$x''(t) = f(t, x(t), x'(t)), \quad x(a) = 0, \quad x(b) = B.$$

By Corollary (1.7.8), we have

$$x'(t) = x'(a) + \int_a^t x''(s)ds = x'(a) + \int_a^t f(s, x(s), x'(s))ds.$$

Thus

$$x'(a) = \frac{1}{b-a}\left\{B - \int_a^b \left[\int_a^t f(s, x(s), x'(s))ds\right] dt\right\}.$$

Since

$$B_1(\alpha) - \int_a^b \left[\int_a^t f_1(s, x(s), x'(s), \alpha)ds\right] dt \quad = \quad \int_a^b \left\{\frac{B_1(\alpha)}{b-a} - \int_a^t f_1(s, x(s), x'(s), \alpha)ds\right\} dt$$

$$= \int_a^b \left\{\int_a^t \left[\frac{B_1(\alpha)}{(b-a)(t-a)} - f_1(s, x(s), x'(s), \alpha)\right] ds\right\} dt,$$

and

$$\frac{\partial}{\partial\alpha}\left[\frac{B_1(\alpha)}{(b-a)(t-a)} - f_1(s, x(s), x'(s), \alpha)\right] = \frac{1}{(b-a)(t-a)}\frac{dB_1(\alpha)}{d\alpha} -$$

$$\frac{\partial}{\partial\alpha} f_1(s, x(s), x'(s), \alpha) \geq \frac{1}{(b-a)^2}\frac{dB_1(\alpha)}{d\alpha} - \frac{\partial}{\partial\alpha} f_1(s, x(s), x'(s), \alpha)$$

$$\geq 0, \text{ a.e. on } [0,1] \ (a < t < b, a \leq s \leq t),$$

we have $\dfrac{B_1(\alpha)}{(b-a)(t-a)} - f_1(s, x(s), x'(s), \alpha)$ is monotone increasing on $[0, 1]$ with respect

to α, therefore $B_1(\alpha) - \int\limits_a^b \left[\int\limits_a^t f_1(s, x(s), x'(s), \alpha) ds \right] dt$ is monotone increasing on $[0, 1]$.

Similarly, $B_2(\alpha) - \int\limits_a^b \left[\int\limits_a^t f_2(s, x(s), x'(s), \alpha) ds \right] dt$ is monotone decreasing on $[0, 1]$.

By Definition (1.1.3), $B - \int\limits_a^b \left[\int\limits_a^t f(s, x(s), x'(s), \alpha) ds \right] dt \in R_F$, thus, $x'(t) \in R_F$, $t \in T$,

and then $x(t) = \int\limits_a^t x'(s) ds \in R_F$, $t \in T$. That is to say $x(t)$ is continuously derivable.

By Proposition (1.7.3) and Theorem (1.7.9), we have

$$x(t) + \int\limits_a^b G(t,s) f(s, x(s), x'(s)) ds = x(t) + \int\limits_a^b G(t,s) x''(s) ds$$

$$= x(t) + \frac{b-t}{b-a} \int\limits_a^t (s-a) x''(s) ds + \frac{t-a}{b-a} \int\limits_t^b (b-s) x''(s) ds$$

$$= x(t) + \frac{b-t}{b-a} \left[(s-a) x'(s) \mid_a^t - \int\limits_a^t x'(s) ds \right] + \frac{t-a}{b-a} \left[(b-s) x'(s) \mid_t^b + \int\limits_t^b x'(s) ds \right]$$

$$= x(t) - x(t) + \frac{B(t-a)}{b-a} = w(t),\ t \in T,$$

thus, $x(t)$ is a solution of (2.5).

Let $x(t)$ be a solution of (2.5), then

$$x(t) + \int\limits_a^b G(t,s) f(s, x(s), x'(s)) ds = w(t).$$

By Proposition (1.7.3)

$$x(t) + \frac{b-t}{b-a} \int\limits_a^t (s-a) f(s, x(s), x'(s)) ds + \frac{t-a}{b-a} \int\limits_t^b (b-s) f(s, x(s), x'(s)) ds = w(t), \quad (2.6)$$

hence $x(a) = w(a) = 0$, $x(b) = w(b) = B$.

Since

$$\frac{B_1(\alpha)(t-a)}{b-a} - \frac{b-t}{b-a}\int_a^t (s-a) f_1(s,x(s),x'(s),\alpha)ds -$$

$$\frac{t-a}{b-a}\int_t^b (b-s) f_1(s,x(s),x'(s),\alpha)ds$$

$$= \int_a^t \left[\frac{t-a}{(b-a)^2}B_1(\alpha) - \frac{b-t}{b-a}(s-a)f_1(s,x(s),x'(s),\alpha)\right]ds$$

$$+\int_t^b \left[\frac{t-a}{(b-a)^2}B_1(\alpha) - \frac{t-a}{b-a}(b-s)f_1(s,x(s),x'(s),\alpha)\right]ds,$$

and

$$\frac{\partial}{\partial\alpha}\left[\frac{t-a}{(b-a)^2}B_1(\alpha) - \frac{b-t}{b-a}(s-a)f_1(s,x(s),x'(s),\alpha)\right] =$$

$$\frac{t-a}{(b-a)^2}\frac{dB_1(\alpha)}{d\alpha} - \frac{b-t}{b-a}(s-a)\times\frac{\partial}{\partial\alpha}f_1(s,x(s),x'(s),\alpha) \geq$$

$$(t-a)\left[\frac{1}{(b-a)^2}\frac{dB_1(\alpha)}{d\alpha} - \frac{\partial}{\partial\alpha}f_1(s,x(s),x'(s),\alpha)\right] \geq 0, \text{a.e. on } [0,1]\ (a \leq s \leq t),$$

$$\frac{\partial}{\partial\alpha}\left[\frac{t-a}{(b-a)^2}B_1(\alpha) - \frac{t-a}{b-a}(b-s)f_1(s,x(s),x'(s),\alpha)\right] =$$

$$\frac{t-a}{(b-a)^2}\frac{dB_1(\alpha)}{d\alpha} - \frac{t-a}{b-a}(b-s)\times\frac{\partial}{\partial\alpha}f_1(s,x(s),x'(s),\alpha) \geq$$

$$(t-a)\left[\frac{1}{(b-a)^2}\frac{dB_1(\alpha)}{d\alpha}-\frac{\partial}{\partial\alpha}f_1(s,x(s),x'(s),\alpha)\right]\geq 0, \text{a.e. on } [0,1]\ (t\leq s\leq b),$$

we have $\dfrac{B_1(\alpha)(t-a)}{b-a}-\displaystyle\int_a^b G(t,s)f_1(s,x(s),x'(s),\alpha)ds$ is monotone increasing with respect to $\alpha\in[0,1]$. Similarly, $\dfrac{B_2(\alpha)(t-a)}{b-a}-\displaystyle\int_a^b G(t,s)f_2(s,x(s),x'(s),\alpha)ds$ is monotone decreasing with respect to $\alpha\in[0,1]$.

Hence, by Definition (1.1.3), $x(t)=w(t)-\displaystyle\int_a^b G(t,s)f(s,x(s),x'(s))ds\in R_F$, $t\in T$.

By Theorem (1.7.7) and Theorem (1.7.10), differentiating (2.6), we have

$$x'(t)=w'(t)-\frac{d}{dt}\left[\frac{b-t}{b-a}\int_a^t (s-a)f(s,x(s),x'(s))ds\right]-$$

$$\frac{d}{dt}\left[\frac{t-a}{b-a}\int_t^b (b-s)f(s,x(s),x'(s))ds\right]$$

$$=\frac{B}{b-a}-\frac{1}{b-a}\int_a^b (b-s)f(s,x(s),x'(s))ds+\int_a^t f(s,x(s),x'(s))ds$$

$$=\frac{1}{b-a}\int_a^b\left[\frac{B}{b-a}-(b-s)f(s,x(s),x'(s))\right]ds+\int_a^t f(s,x(s),x'(s))ds. \qquad (2.7)$$

Similarly, we have $\dfrac{B}{b-a}-\dfrac{1}{b-a}\displaystyle\int_a^b (b-s)f(s,x(s),x'(s))ds\in R_F$, hence $x'(t)\in R_F$, $t\in T$.

By Theorem (1.7.7), differentiating (2.7), we have $x''(t)=f(t,x(t),x'(t))\in R_F$, $t\in T$, and then $x(t)$ has continuous second derivative and satisfies (2.4), i.e., $x(t)$ is a

solution of (2.4).

So far, we get that solving the boundary value problem of a fuzzy differential equation (2.4) is equivalent to solve a fuzzy integral equation (2.5) under certain conditions. Throughout the proof, although some operations are executed outside R_F, $x(t)$, $x'(t)$, and $x''(t)$ are always inside R_F, which is the advantage that R_F is taken as a closed convex cone in Banach space $X = C[0,1] \times C[0,1]$ and the operations are executed inside X.

Example 2.2.4 [2] In problems (2.4) and (2.5), take $T = [0,1]$ and $f(t,x,x') = B = [\alpha, 2-\alpha]$, $\alpha \in [0,1]$.
We can see that f, B satisfy the conditions of Theorem (2.2.3), thus Theorem (2.2.3) holds.

Lemma 2.2.5 [2] Let K, L be two positive real numbers in linear space $C^1(T,X)$, and define the norm by

$$\|x\|_{\infty} = K.\max_{t\in T}\|x(t)\| + L.\max_{t\in T}\|x'(t)\|,$$

then $C^1(T,X)$ is a Banach space and $C^1(T,R_F)$ is a closed convex cone in $C^1(T,X)$.

Theorem 2.2.6 [2] Let $f \in C(T \times R_F \times R_F, R_F)$ and $B \in R_F$ satisfy the conditions of Theorem (2.2.3), and there exist K, $L > 0$ such that $\forall (u,v),(x,y) \in R_F \times R_F$,

$$\|f(t,u,v) - f(t,x,y)\| \leq K.\|u-x\| + L.\|v-y\| \qquad (t \in T),$$

where $r = K.\dfrac{(b-a)^2}{8} + L.\dfrac{b-a}{2} < 1$, then problem (2.4) has a unique solution in $C^2(T,R_F)$.

Proof. By Lemma (2.2.5), $C^1(T,R_F)$ is a closed convex cone in Banach space $C^1(T,X)$.

Let $F : C^1(T, R_F) \longrightarrow C^1(T, R_F)$ be

$$(Fu)(t) = w(t) + \int_a^b G^*(t,s) f(s, u(s), u'(s)) ds, u \in C^1(T, R_F),\ t \in T,$$

where $G^*(t,s) = -G(t,s)$. Then it follows from the proof of Theorem (2.2.3) that $(Fu)(t) \in R_F$, $(Fu)'(t) \in R_F$, $(Fu)''(t) \in R_F$, so $Fu \in C^1(T, R_F)$. $\forall u, x \in R_F$, we have

$$\|Fu - Fx\|_\infty = K.\max_{t\in T} \|(Fu)(t) - (Fx)(t)\| + L.\max_{t\in T} \|(Fu)'(t) - (Fx)'(t)\|$$

and

$$\begin{aligned} \|(Fu)(t) - (Fx)(t)\| &\leq \int_a^b |G^*(t,s)| \left[K. \|u(s) - x(s)\| + L. \|u'(s) - x'(s)\|\right] ds \\ &\leq \|u - x\|_\infty \int_a^b |G(t,s)|\, ds \leq \frac{(b-a)^2}{8} \|u - x\|_\infty . \end{aligned}$$

Obviously $\forall u \in C^1(T, R_F)$ by Proposition (1.7.3), Theorems (1.7.10) and (1.7.7) we have

$$\begin{aligned} (Fu)'(t) &= w'(t) + \int_a^t \frac{s-a}{b-a} f(s, u(s), u'(s)) ds + \int_t^b \frac{s-b}{b-a} f(s, u(s), u'(s)) ds \\ &= w'(t) + \int_a^b \frac{\partial G^*(t,s)}{\partial t} f(s, u(s), u'(s)) ds, \end{aligned}$$

hence

$$\begin{aligned} \|(Fu)'(t) - (Fx)'(t)\| &\leq \int_a^b \left|\frac{\partial G^*(t,s)}{\partial t}\right| \left[K. \|u(s) - x(s)\| + L. \|u'(s) - x'(s)\|\right] ds \\ &\leq \|u - x\|_\infty \int_a^b \left|\frac{\partial G(t,s)}{\partial t}\right| ds \leq \frac{b-a}{2} \|u - x\|_\infty . \end{aligned}$$

And thus,

$$\|Fu - Fx\|_{\infty} \le K.\frac{(b-a)^2}{8}\|u - x\|_{\infty} + L.\frac{b-a}{2}\|u - x\|_{\infty} = r.\|u - x\|_{\infty},$$

i.e., F is a contraction mapping. So there exists a unique $x \in C^1(T, R_{\mathcal{F}})$ such that $Fx = x$, and further by Theorem (2.2.3) problem (2.4) has a unique solution $x(t)$, $t \in T$, in $C^2(T, R_{\mathcal{F}})$.

The example below shows that the equivalence between the equations (2.4) and (2.5) does not hold.

Example 2.2.7 [3] We consider the two point boundary value problem

$$\left.\begin{array}{c} x''(t) = f(t, x(t), x'(t)), \\ x(a) = \tilde{0}, x(b) = \tilde{0}, \end{array}\right\} \tag{2.8}$$

(here $\tilde{0} = \chi_{\{0\}} \in R_{\mathcal{F}}$) and the integral equation

$$x(t) = \int_a^b G(t,s) f(s, x(s), x'(s)) ds, \tag{2.9}$$

where $G(t, s)$ is a Green's function. Let $f : [0, 1] \times R_{\mathcal{F}} \times R_{\mathcal{F}} \longrightarrow R_{\mathcal{F}}$, $f(t, x, x') = (0, 1, 2)$, where $(0, 1, 2)$ is the triangular fuzzy number having the endpoints of the $\alpha-$level sets $\underline{f}_{\alpha} = \alpha$ and $\overline{f}_{\alpha} = 2 - \alpha$. It is well known that Green's function on $[0, 1]$ interval is

$$G(t,s) = \begin{cases} -s(1-t), s \le t, \\ -t(1-s), s > t. \end{cases}$$

Let us compute the solution of Eq. (2.9). Since $[(-1).u]_{\alpha} = [-\overline{u}_{\alpha}, -\underline{u}_{\alpha}]$ and since we

can integrate the terms levelwise, we have

$$\begin{aligned}
x(t) &= \int_0^t G(t,s)[\alpha, 2-\alpha]ds + \int_t^1 G(t,s)[\alpha, 2-\alpha]ds \\
&= \int_0^t (-s(1-t))[\alpha, 2-\alpha]ds + \int_t^1 (-t(1-s))[\alpha, 2-\alpha]ds \\
&= \int_0^t [-s(1-t)(2-\alpha), -\alpha s(1-t)]ds + \int_t^1 [-t(1-s)(2-\alpha), -\alpha t(1-s)]ds \\
&= \left[\frac{2-\alpha}{2}(t^2-t), \frac{\alpha}{2}(t^2-t)\right].
\end{aligned}$$

Let us suppose that this function is a solution of (2.8). Then we observe that

$$\begin{aligned}
\underline{x(t)}_\alpha &= \frac{2-\alpha}{2}(t^2-t), \\
\overline{x(t)}_\alpha &= \frac{\alpha}{2}(t^2-t),
\end{aligned}$$

and we have

$$\begin{aligned}
\underline{x'(t)}_\alpha &= \frac{2-\alpha}{2}(2t-1), \\
\overline{x'(t)}_\alpha &= \frac{\alpha}{2}(2t-1).
\end{aligned}$$

Then $x'(t) = \left[\frac{2-\alpha}{2}(2t-1), \frac{\alpha}{2}(2t-1)\right]$ is a fuzzy number only for $t \leq \frac{1}{2}$. Moreover, we have $\underline{x''(t)}_\alpha = 2-\alpha$ and $\overline{x''(t)}_\alpha = \alpha$. These values define a fuzzy number if and only if $2-\alpha < \alpha$, i.e., $2 < 2\alpha$, $\forall \alpha \in [0,1]$, which is a contradiction. This proves that a solution of (2.9) is not necessarily a solution of (2.8) under Hukuhara differentiability, so these equations are not equivalent.

Chapter Three
Reproducing Kernel Hilbert Space Method

In this chapter, we will give the presentation of the exact solutions and approximate solutions for the fuzzy boundary value problems in the reproducing kernel space.

The method has the following advantages: Firstly, the conditions for determining solution of equations (3.11) can be imposed on the reproducing kernel space and therefore reproducing kernel satisfying the conditions for determining solution can be calculated. We will use the kernel to solve problems. Secondly, the iterative sequences $u_n(x)$, $v_n(x)$ of approximate solutions converge in $C^2[0,1]$ to the solutions $u(x)$, $v(x)$ respectively.

3.1 Basic Concepts of the Reproducing Kernel Hilbert Space

In this section, we make a survey of the theory of reproducing kernel Hilbert space associated with positive definite kernels.

Definition 3.1.1 [13] A function $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ is a kernel if:

(i) K is symmetric: $K(x, y) = K(y, x)$.

(ii) K is positive semi-definite, i.e., $\forall x_1, x_2, ..., x_n \in \mathcal{X}$, the "Gram Matrix" K defined by $K_{ij} = K(x_i, x_j)$ is positive semi-definite.

Definition 3.1.2 [13] Let H be a Hilbert space of functions on a set $\mathcal{X}$. Denote by $\langle f, g\rangle$ the inner product and let $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$ be the norm in H, for f and $g \in H$. The complex valued function $K(y, x)$ of y and x in $\mathcal{X}$ is called a reproducing kernel of H if the following are satisfied:

(i) For every x, $K_x(y) = K(y, x)$ as a function of y belongs to H.

(ii) The reproducing property: for every $x \in \mathcal{X}$ and every $f \in H$,

$$f(x) = \langle f(.), K_x(.) \rangle . \tag{3.1}$$

Applying (3.1) to the function K_x at y, we get

$$K_x(y) = \langle K_x, K_y \rangle , \text{ for } x, y \in \mathcal{X}.$$

By the above relation and by (3.1), for every $x \in \mathcal{X}$, we obtain

$$\|K_x\| = \langle K_x, K_x \rangle^{\frac{1}{2}} = K(x,x)^{\frac{1}{2}}.$$

Definition 3.1.3 [13] A Hilbert space H of functions on a set $\mathcal{X}$ is called a reproducing kernel Hilbert space (Sometimes abbreviated by RKHS) if there exists a reproducing kernel K of H such that K spans H, i.e., every $f \in H$ can be written as: $f(.) = \sum \alpha_i K(., x_i)$, where $\alpha_i \in \mathbb{C}$ and $x_i \in \mathcal{X}$.

Theorem 3.1.4 (Aronszajn, 1950) If a Hilbert space H of functions on a set $\mathcal{X}$ admits a reproducing kernel, then the reproducing kernel $K(x,y)$ is uniquely determined by the Hilbert space H.

Proof. Let $K(y,x)$ be a reproducing kernel of H. Suppose that there exists another kernel $R(y,x)$ of H. Then, for all $x \in \mathcal{X}$, applying the reproducing property for K and R, we get

$$\begin{aligned} \|K_x - R_x\|^2 &= \langle K_x - R_x, K_x - R_x \rangle = \langle K_x - R_x, K_x \rangle - \langle K_x - R_x, R_x \rangle \\ &= (K_x - R_x)(x) - (K_x - R_x)(x) = 0. \end{aligned}$$

Hence $K_x = R_x$, that is, $K_x(y) = R_x(y)$ for all $y \in \mathcal{X}$. This means that $K(y,x) = R(y,x)$ for all $x, y \in \mathcal{X}$. The proof is complete.

Remark 3.1.5 [13] Let H be a RKHS and its kernel $K(y,x)$ on $\mathcal{X}$, then, for all $x, y \in \mathcal{X}$, we have

(i) $K(y,y) \geq 0$.

(ii) $K(y,x) = \overline{K(x,y)}$.

(iii) $|K(y,x)|^2 \leq K(x,x)\, K(y,y)$, (Schwarz Inequality).

(iv) Let $x_0 \in X$. Then the followings are equivalent:

  (a) $K(x_0, x_0) = 0$.

  (b) $K(y, x_0) = 0$ for all $y \in \mathcal{X}$.

  (c) $f(x_0) = 0$ for all $f \in H$.

3.2 Reproducing Kernel Functions Represented By Form of Polynomials

In this section, we will find out the expression forms of the reproducing kernel functions in the Sobolev space $W_2^m[a,b]$. These expressions can be represented by piecewise polynomials of degree $2m-1$.

The function space W_2^m is defined as follows:
$W_2^m[a,b] = \{f(x) \mid f^{(m-1)}(x)$ is absolutely continuous, $f^{(m)}(x) \in L_2[a,b]$, $f(a) = f(b) = 0$, $x \in [a,b]\}$.

The inner product and the norm in the function space $W_2^m[a,b]$ are defined as follows respectively:
for any functions $f(x)$, $g(x) \in W_2^m[a,b]$,

$$\langle f, g\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} f^{(i)}(a) g^{(i)}(a) + \int_a^b f^{(m)}(x) g^{(m)}(x) dx, \tag{3.2}$$

$$\|f\|_{W_2^m[a,b]} = \sqrt{\langle f, f\rangle_{W_2^m[a,b]}}\ . \tag{3.3}$$

It is easy to prove that $W_2^m[a,b]$ is an inner space with the definition (3.2).

Theorem 3.2.1 [11] Function space $W_2^m[a,b]$ is a Hilbert space.

Lemma 3.2.2 [11] $W_2^m[a,b]$ is a reproducing kernel space if and only if for any $x \in [a,b]$, $I : f \longrightarrow f(x)$ is a bounded functional in $W_2^m[a,b]$.

Theorem 3.2.3 [11] Function space $W_2^m[a,b]$ is a reproducing kernel space.

Now, let's find out the expression form of the reproducing kernel function $K_x(y)$ in the space $W_2^m[a,b]$.

Suppose $K_x(y)$ is the reproducing kernel function of the space $W_2^m[a,b]$, then for each fixed $x \in [a,b]$ and any $u(y) \in W_2^m[a,b]$, $y \in [a,b]$ we have $\langle u(y), K_x(y)\rangle = u(x)$. Based on Equations (3.2) and (3.3), we have

$$\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\, K_x^{(i)}(a) + \int_a^b u^{(m)}(y)\, K_x^{(m)}(y)\, dy. \tag{3.4}$$

Applying the integration by parts for the second scheme of the right-hand of Equation (3.4), we obtain

$$\begin{aligned} \int_a^b u^{(m)}(y)\, K_x^{(m)}(y)\, dy &= \sum_{i=0}^{m-1} (-1)^i\, u^{(m-i-1)}(y)\, K_x^{(m+i)}(y) \,\Big|_{y=a}^{b} \\ &\quad + \int_a^b (-1)^m\, u(y)\, K_x^{(2m)}(y)\, dy. \end{aligned}$$

Let $j = m-i-1$, the first term of the right-hand side of the above formula can be rewritten as

$$\sum_{i=0}^{m-1} (-1)^i\, u^{(m-i-1)}(y)\, K_x^{(m+i)}(y) \,\Big|_{y=a}^{b} = \sum_{j=0}^{m-1} (-1)^{m-j-1}\, u^{(j)}(y)\, K_x^{(2m-j-1)}(y) \,\Big|_{y=a}^{b}.$$

After some simplification, Equation (3.4) became's

$$\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\left(K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a)\right) +$$

$$\sum_{i=0}^{m-1} (-1)^{m-i-1} u^{(i)}(b)\, K_x^{(2m-i-1)}(b) + \int_a^b (-1)^m u(y)\, K_x^{(2m)}(y)\, dy.$$

Since $K_x(y), u(y) \in W_2^m[a,b]$, it follows that

$$K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a) = 0, K_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1.$$

Then $\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} = \int_a^b u(y)\left((-1)^m K_x^{(2m)}(y)\right) dy.$

Now, for each $x \in [a,b]$, if $K_x(y)$ satisfies $(-1)^m K_x^{(2m)}(y) = \delta(x-y)$, where δ is dirac-delta function, then

$$\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} = \int_a^b u(y)\,\delta(x-y)\, dy = u(x).$$

Obviously, $K_x(y)$ is the reproducing kernel of the space $W_2^m[a,b]$.

Therefore, $K_x(y)$ is the solution of the following generalized differential equations:

$$\begin{cases} (-1)^m K_x^{(2m)}(y) = \delta(x-y) \\ K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a) = 0, i = 0, 1, ..., m-1 \\ K_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1. \end{cases} \tag{3.5}$$

While $x \neq y$

$$(-1)^m K_x^{(2m)}(y) = 0. \tag{3.6}$$

with the boundary conditions (BC's):

$$K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a) = 0, K_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1. \tag{3.7}$$

The characteristic equation of Equation (3.6) is $\lambda^{2m} = 0$, and their characteristic values are $\lambda = 0$ with $2m$ multiple roots. So, the general solution of Equation (3.6) is as follows:

$$K_x(y) = \begin{cases} \sum\limits_{i=0}^{2m-1} p_i(x) y^i, & y \leq x; \\ \sum\limits_{i=0}^{2m-1} q_i(x) y^i, & y > x. \end{cases} \tag{3.8}$$

On the other hand, since $(-1)^m K_x^{(2m)}(y) = \delta(x-y)$, we have

$$K_x^{(i)}(x+0) = K_x^{(i)}(x-0), \ i = 0, 1, ..., 2m-2. \tag{3.9}$$

Integrating $(-1)^m K_x^{(2m)}(y) = \delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $K_x^{(2m-1)}(y)$ at $y = x$ given by

$$(-1)^m \left(K_x^{(2m-1)}(x+0) - K_x^{(2m-1)}(x-0)\right) = 1. \tag{3.10}$$

Equations (3.9) and (3.10) provided $2m$ conditions for solving the coefficients $p_i(x)$ and $q_i(x)$, $i = 0, 1, ..., 2m-1$, in Equation (3.8). Further, Equation (3.7) provided $2m$ BC's. So, we have $4m$ equations. It is easy to know that these $4m$ equations are linear equations with the variables $p_i(x)$ and $q_i(x)$, and the unknown coefficients $p_i(x)$ and $q_i(x)$ of Equation (3.8) could be calculated out by many methods such as Green's function method or by using Mathematica 7.0 software package.

As long as the coefficients $p_i(x)$ and $q_i(x)$ are known, the exact expression of the producing kernel function $K_x(y)$ of the space $W_2^m[a,b]$ could be calculated out from Equation (3.8). The expression of $K_x(y)$ is a piecewise polynomial with $2m-1$ degree.

The following corollary summarized some important properties of the reproducing kernel $K_x(y)$ and it is easily obtained.

Corollary 3.2.4 [42]The reproducing kernel $K_x(y)$ is symmetric, unique and $K_x(x) \geq 0$, for any fixed $x \in [a,b]$.

Proof. By the reproducing property, we have

$$K_x(y) = \langle K_x(\cdot), K_y(\cdot)\rangle = \langle K_y(\cdot), K_x(\cdot)\rangle \ = K_y(x).$$

Now, let $K_x(y)$ and $R_x(y)$ be all the reproducing kernels of the space $W_2^m[a,b]$,then $K_x(y) = \langle K_x(\cdot), R_y(\cdot)\rangle = \langle R_y(\cdot), K_x(\cdot)\rangle = R_y(x)$. By the symmetry of $R_x(y)$, we have the unique representation of $K_x(y)$. For the last condition, we note that $K_x(x) = \langle K_x(\cdot), K_x(\cdot)\rangle = \|K_x(\cdot)\|^2 \geq 0$.

Before we start in the representation of expressions of such reproducing kernel functions, we need the following remarks:

(i) If the functions in space $W_2^m[a,b]$ require more special BC's, *e.g.*, the BC's of the second-order differential equations as follows: $u(a) = \alpha$, $u(b) = \beta$, or $u'(a) = \alpha$, $u'(b) = \beta$; or the linear BC's: $a_1u(a) + b_1u'(a) = \alpha$, $a_2u(b) + b_2u'(b) = \beta$; or the periodic linear BC's: $u(a) = u(b)$, $u'(a) = u'(b)$. These different kinds of BC's could be contained in space $W_2^m[a,b]$ after homogenization, i.e., we can find the reproducing kernel function $K_x(y)$ which satisfies these BC's.

(ii) If we are re-defining the inner product in Equation (3.2) of the space $W_2^m[a,b]$ then the unique reproducing kernel function of that space can be represented (not necessary polynomial) by using the same procedure above.

We are ready to present some expressions of reproducing kernel function in the space $W_2^m[0,1]$ by using the approaches proposed in this section.

1. Reproducing kernel function $K(x,y)$ in $W_2^1[0,1]$:

$$K(x,y) = \begin{cases} 1+y, & y \leq x, \\ 1+x, & y > x, \end{cases}$$

where x, $y \in [0,1]$.

2. Reproducing kernel function $K(x,y)$ in $W_2^2[0,1]$:

$$K(x,y)=\begin{cases}1-\dfrac{y^3}{6}+\dfrac{1}{2}xy(2+y), & y\le x,\\[2ex] 1-\dfrac{x^3}{6}+\dfrac{1}{2}xy(2+x), & y>x,\end{cases}$$

where $x,\ y\in[0,1]$.

3.3 Description of Reproducing Kernel Method

In this section, we present a new algorithm to solve a class of boundary value problems in the reproducing kernel space $W_2^3[0,1]$. The algorithm is efficiently applied to solving some model problems with the comparison between the numerical solutions and the exact solutions.

Assume that the system of second-order differential equations

$$\left.\begin{array}{c} u''(x)=f(x,u(x),u'(x),v(x),v'(x)),\quad 0<x<1,\\ v''(x)=g(x,u(x),u'(x),v(x),v'(x)),\quad 0<x<1,\\ u(0)=0,\ u(1)=0,\\ v(0)=0,\ v(1)=0. \end{array}\right\} \tag{3.11}$$

Let $Lu=u''$, $L: W_2^3[0,1]\longrightarrow W_2^1[0,1]$, where $W_2^3[0,1]=\{u(x)\mid u''(x)$ is absolutely continuous real value function, $u'''(x)\in L_2[0,1]$, $u(0)=u(1)=0\}$ with inner product in $W_2^3[0,1]$ given by

$$\langle u(x),v(x)\rangle_{W_2^3[0,1]}=\sum_{i=0}^{2}u^{(i)}(0)\,v^{(i)}(0)+\int_0^1 u'''(x)\,v'''(x)\,dx.$$

Then Eqs. (3.11) can be converted into the form as follows

$$\left.\begin{array}{c} Lu(x)=f(x,u(x),u'(x),v(x),v'(x)),\quad 0<x<1,\\ Lv(x)=g(x,u(x),u'(x),v(x),v'(x)),\quad 0<x<1, \end{array}\right\} \tag{3.12}$$

where $u(x)$, $v(x) \in W_2^3[0,1]$ and f, $g \in W_2^1[0,1]$. It is easy to prove that L is a bounded linear operator.

Now, we construct an orthogonal function system $\{\psi_i(x)\}_{i=1}^{\infty}$ of the space $W_2^3[0,1]$. For a countable dense set $\{x_i\}_{i=1}^{\infty}$ of $[0,1]$, let $\varphi_i(x) = K_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. So, from the properties of reproducing kernel $K_x(y)$, for every $u(x) \in W_2^1[0,1]$, it follows that $\langle u(x), \varphi_i(x)\rangle_{W_2^1} = \langle u(x), K_{x_i}(x)\rangle_{W_2^1}$ $= u(x_i)$. Obviously, $\psi_i(x) \in W_2^3[0,1]$. In terms of the properties of $K_x(y)$, one obtains

$$\langle u(x), \psi_i(x)\rangle_{W_2^3} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i),$$

where $i = 1, 2, ...$

We want to give important lemmas to construct the RKHS method.

Lemma 3.3.1 [20] If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system of $W_2^3[0,1]$ if L^{-1} in Eqs. (3.12) existent.

Proof. For each fixed $u(x) \in W_2^3[0,1]$.
If $\langle u(x), \psi_i(x)\rangle_{W_2^3} = 0$, $i = 1, 2, ...$, then
$\langle u(x), \psi_i(x)\rangle_{W_2^3} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i) = 0.$
Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore, $Lu(x) = 0$. It follows that $u(x) = 0$ from the existence of L^{-1} and the continuity of $u(x)$. The proof is complete.

Lemma 3.3.2 [27] If $Lu(x) = f(x)$, $0 \le x \le 1$, $u(0) = u(1) = 0$, where $u \in W_2^3[0,1]$, $f \in W_2^1[0,1]$. If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\psi_i(x) = L_y K_x(y)\,|_{y=x_i}$.

Proof. We have

$$\begin{aligned}\psi_i(x) &= L^*\varphi_i(x) = \langle L^*\varphi_i(x),\ K_x(y)\rangle_{W_2^3[0,1]} \\ &= \langle \varphi_i(x), LK_x(y)\rangle_{W_2^1[0,1]} = L_y K_x(y)\big|_{y=x_i}.\end{aligned}$$

Lemma 3.3.3 [21] $\psi_i(0) = \psi_i(1) = 0$, $i = 1, 2, ...$

Proof. $\psi_i(0) = \langle \psi_i(y), K_0(y) \rangle_{W_2^3} = \langle L^* \varphi_i(y), K_0(y) \rangle_{W_2^3} = \langle \varphi_i(y), L_y K_0(y) \rangle_{W_2^1}$. By the symmetry of $K_0(y)$, we arrive at $K_0(y) = K_y(0) = 0$, thus $\psi_i(0) = 0$. Similarly, we further obtain $\psi_i(1) = 0$.

The orthonormal function system $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$ of the space $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \ i = 1, 2, ... \tag{3.13}$$

where β_{ik} are orthogonalization coefficients and are given by

$$\beta_{11} = \frac{1}{\|\psi_1\|}, \quad \beta_{ij} = \frac{-\sum\limits_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum\limits_{k=1}^{i-1} c_{ik}^2}}, (j < i), \quad \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum\limits_{k=1}^{i-1} c_{ik}^2}}, \tag{3.14}$$

in which $c_{ik} = \left\langle \psi_i, \overline{\psi}_k \right\rangle_{W_2^3}$ and $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$ is the orthonormal system in the space $W_2^3[0,1]$.

Lemma 3.3.4 [17] If $u(x)) \in W_2^3[0,1]$, then there exists $M > 0$, such that $\|u\|_{C^2[0,1]} \leq \|u\|_{W_2^3[0,1]}$, where

$$\|u\|_{C^2[0,1]} = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)| + \max_{x\in[0,1]} |u''(x)|.$$

Lemma 3.3.5 [17] If $\|u_n - u\|_{W_2^3} \longrightarrow 0$, $\|v_n - v\|_{W_2^3} \longrightarrow 0$, $x_n \longrightarrow x$, $(n \longrightarrow \infty)$ and $f(x,y,z,w,v)$, $g(x,y,z,w,v)$ for $x \in [0,1]$, y, z, w, $v \in (-\infty, \infty)$ are continuous with respect to x, y, z, w, v, then

$$\begin{aligned} f(x_n, u_{n-1}(x_n), u'_{n-1}(x_n), v_{n-1}(x_n), v'_{n-1}(x_n)) &\longrightarrow f(x, u(x), u'(x), v(x), v'(x)) \text{ as } n \longrightarrow \infty, \\ g(x_n, u_{n-1}(x_n), u'_{n-1}(x_n), v_{n-1}(x_n), v'_{n-1}(x_n)) &\longrightarrow g(x, u(x), u'(x), v(x), v'(x)) \text{ as } n \longrightarrow \infty. \end{aligned}$$

In the next theorem, we will give the exact solution of Eqs. (3.11) in the reproducing kernel space.

Theorem 3.3.6 [17] If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and $u(x)$, $v(x) \in W_2^3[0,1]$ are the solutions of Eqs. (3.12), then $u(x)$, $v(x)$ satisfy the following form, respectively

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k), u'(x_k), v(x_k), v'(x_k))\overline{\psi}_i(x),$$

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k), u'(x_k), v(x_k), v'(x_k))\overline{\psi}_i(x).$$

Proof. $u(x)$ can be expanded to Fourier series in terms of orthonormal basis $\overline{\psi}_i(x)$ in $W_2^3[0,1]$.

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \overline{\psi}_i(x)\right\rangle_{W_2^3} \overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x), \psi_k(x)\right\rangle_{W_2^3} \overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x), L^*\varphi_k(x)\right\rangle_{W_2^3} \overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle Lu(x), \varphi_k(x)\right\rangle_{W_2^1} \overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle f(x, u(x), u'(x), v(x), v'(x)), \varphi_k(x)\right\rangle_{W_2^1} \overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k), u'(x_k), v(x_k), v'(x_k))\overline{\psi}_i(x).
\end{aligned}$$

In the same way, we can get $v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k), u'(x_k), v(x_k), v'(x_k))\overline{\psi}_i(x)$. And the approximate solution can be obtained by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k, u_{k-1}(x_k), u'_{k-1}(x_k), v_{k-1}(x_k), v'_{k-1}(x_k))\overline{\psi}_i(x), \tag{3.15}$$

$$v_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}g(x_k, u_{k-1}(x_k), u'_{k-1}(x_k), v_{k-1}(x_k), v'_{k-1}(x_k))\overline{\psi}_i(x), \tag{3.16}$$

where $u_0(x) = 0$, $v_0(x) = 0$ such that $u_0(x)$, $v_0(x)$ (Fixed) $\in W_2^3\,[0,1]$.

Theorem 3.3.7 [17] Suppose the following conditions are satisfied:

(i) $\|u_n\|_{W_2^3}$, $\|v_n\|_{W_2^3}$ are bounded.

(ii) $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$.

(iii) $f(x, y(x), z(x), w(x), v(x))$, $g(x, y(x), z(x), w(x), v(x)) \in W_2^1\,[0,1]$ for any $y(x)$, $z(x)$, $w(x)$,$v(x) \in W_2^3\,[0,1]$.

Then $u_n(x)$, $v_n(x)$ in iterative formulas (3.15) and (3.16) are convergent to the exact solutions $u(x)$, $v(x)$ of Eqs. (3.12) in $W_2^3\,[0,1]$ and $u(x) = \sum_{i=1}^{\infty} A_i\,\overline{\psi}_i$, $v(x) = \sum_{i=1}^{\infty} B_i\,\overline{\psi}_i$, where

$$\begin{aligned} A_i &= \sum_{k=1}^{i}\beta_{ik}f(x_k, u_{k-1}(x_k), u'_{k-1}(x_k), v_{k-1}(x_k), v'_{k-1}(x_k)), \\ B_i &= \sum_{k=1}^{i}\beta_{ik}g(x_k, u_{k-1}(x_k), u'_{k-1}(x_k), v_{k-1}(x_k), v'_{k-1}(x_k)). \end{aligned}$$

Proof. First, we will prove the convergence of $u_n\,(x)$, $v_n\,(x)$. By Eqs. (3.15) and (3.16), we have

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + A_{n+1}\overline{\psi}_{n+1}\,(x)\,, \\ v_{n+1}(x) &= v_n(x) + B_{n+1}\overline{\psi}_{n+1}\,(x)\,. \end{aligned}$$

From the orthogonality of $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned}
||u_{n+1}||^2_{W_2^3} &= ||u_n||^2_{W_2^3} + (A_{n+1})^2 = ||u_{n-1}||^2_{W_2^3} + (A_n)^2 + (A_{n+1})^2 \\
&\quad ... \\
&= ||u_0||^2_{W_2^3} + \sum_{i=1}^{n+1}(A_i)^2, \\
||v_{n+1}||^2_{W_2^3} &= ||v_n||^2_{W_2^3} + (B_{n+1})^2 = ||v_{n-1}||^2_{W_2^3} + (B_n)^2 + (B_{n+1})^2 \\
&\quad ... \\
&= ||v_0||^2_{W_2^3} + \sum_{i=1}^{n+1}(B_i)^2.
\end{aligned}$$

From boundedness of $||u_n||_{W_2^3}$ and $||v_n||_{W_2^3}$, we have $\sum_{i=1}^{\infty}(A_i)^2 < \infty$, $\sum_{i=1}^{\infty}(B_i)^2 < \infty$, that is, $\{A_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty} \in l^2$ $(i = 1, 2, ...)$.

Let $m > n$, for $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp ... \perp (u_{n+1} - u_n)$, $(v_m - v_{m-1}) \perp (v_{m-1} - v_{m-2}) \perp ... \perp (v_{n+1} - v_n)$, it follows that

$$\begin{aligned}
||u_m(x) - u_n(x)||^2_{W_2^3} &= ||u_m(x) - u_{m-1}(x) + u_{m-1}(x) - ... + u_{n+1}(x) - u_n(x)||^2_{W_2^3} \\
&= ||u_m(x) - u_{m-1}(x)||^2_{W_2^3} + \ ... \ + ||u_{n+1}(x) - u_n(x)||^2_{W_2^3} \\
&= \sum_{i=n+1}^{m}(A_i)^2 \longrightarrow 0, \ (n \longrightarrow \infty) \\
||v_m(x) - v_n(x)||^2_{W_2^3} &= ||v_m(x) - v_{m-1}(x) + v_{m-1}(x) - ... + v_{n+1}(x) - v_n(x)||^2_{W_2^3} \\
&= ||v_m(x) - v_{m-1}(x)||^2_{W_2^3} + \ ... \ + ||v_{n+1}(x) - v_n(x)||^2_{W_2^3} \\
&= \sum_{i=n+1}^{m}(B_i)^2 \longrightarrow 0, \ (n \longrightarrow \infty).
\end{aligned}$$

Considering the completeness of $W_2^3[0,1]$, there exists $u(x), v(x) \in W_2^3[0,1]$ such that $u_n(x) \longrightarrow u(x)$ as $n \to \infty$ in sense of the norm of $W_2^3[0,1]$ and $v_n(x) \longrightarrow v(x)$ as $n \to \infty$ in sense of the norm of $W_2^3[0,1]$.

Second, we will prove that $u(x)$, $v(x)$ are the solutions of Eqs. (3.12).

By Lemma (3.3.4) and since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, we know $u_n(x)$, $v_n(x)$ converge uniformly to $u(x)$, $v(x)$, respectively. It follows that, on taking limits in (3.15) and

(3.16), we have

$$u(x)=\sum_{i=1}^{\infty}A_i\overline{\psi}_i,\quad v(x)=\sum_{i=1}^{\infty}B_i\overline{\psi}_i.$$

Since

$$\begin{aligned}(Lu)(x_j) &= \sum_{i=1}^{\infty}A_i\left\langle L\overline{\psi}_i(x),\varphi_j(x)\right\rangle_{W_2^1}=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(x),L^*\varphi_j(x)\right\rangle_{W_2^3}\\ &= \sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(x),\psi_j(x)\right\rangle_{W_2^3},\end{aligned}$$

and

$$\begin{aligned}(Lv)(x_j) &= \sum_{i=1}^{\infty}B_i\left\langle L\overline{\psi}_i(x),\varphi_j(x)\right\rangle_{W_2^1}=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(x),L^*\varphi_j(x)\right\rangle_{W_2^3}\\ &= \sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(x),\psi_j(x)\right\rangle_{W_2^3}.\end{aligned}$$

It follows that

$$\sum_{j=1}^{n}\beta_{nj}(Lu)(x_j)=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(x),\sum_{j=1}^{n}\beta_{nj}\psi_j(x)\right\rangle_{W_2^3}=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(x),\overline{\psi}_n(x)\right\rangle_{W_2^3}=A_n,$$

and

$$\sum_{j=1}^{n}\beta_{nj}(Lv)(x_j)=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(x),\sum_{j=1}^{n}\beta_{nj}\psi_j(x)\right\rangle_{W_2^3}=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(x),\overline{\psi}_n(x)\right\rangle_{W_2^3}=B_n.$$

If $n=1$, then

$$\begin{aligned}(Lu)(x_1) &= f(x_1,u_0(x_1),u_0'(x_1),v_0(x_1),v_0'(x_1)),\\ (Lv)(x_1) &= g(x_1,u_0(x_1),u_0'(x_1),v_0(x_1),v_0'(x_1)).\end{aligned}$$

If $n = 2$, then

$$\begin{aligned} (Lu)(x_2) &= f(x_2, u_1(x_2), u_1'(x_2), v_1(x_2), v_1'(x_2)), \\ (Lv)(x_2) &= g(x_2, u_1(x_2), u_1'(x_2), v_1(x_2), v_1'(x_2)). \end{aligned}$$

Furthermore, it is easy to see by induction that

$$\begin{aligned} (Lu)(x_j) &= f(x_j, u_{j-1}(x_j), u_{j-1}'(x_j), v_{j-1}(x_j), v_{j-1}'(x_j)), \\ (Lv)(x_j) &= g(x_j, u_{j-1}(x_j), u_{j-1}'(x_j), v_{j-1}(x_j), v_{j-1}'(x_j)). \end{aligned}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, for any $y \in [0, 1]$, there exists subsequence $\{x_{n_j}\}$ such that $x_{n_j} \longrightarrow y$, as $j \longrightarrow \infty$.
Hence, let $j \longrightarrow \infty$ in the last equations, by the convergence of $u_n(x)$, $v_n(x)$ and Lemma (3.3.5), we have

$$\begin{aligned} (Lu)(y) &= f(y, u(y), u'(y), v(y), v'(y)), \\ (Lv)(y) &= g(y, u(y), u'(y), v(y), v'(y)). \end{aligned}$$

That is, $u(x)$, $v(x)$ are the solutions of Eqs. (3.12) and

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i, \quad v(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i.$$

The proof is complete.

From Lemma (3.3.4), we have the following corollary.

Corollary 3.3.8 [17] Assume that the conditions of Theorem (3.3.5) hold, then $u_n(x)$, $v_n(x)$ in (3.15) and (3.16) satisfy

$$\|u_n(x) - u(x)\|_{C^2} \longrightarrow 0, \|v_n(x) - v(x)\|_{C^2} \longrightarrow 0, n \longrightarrow \infty,$$

where $u(x)$, $v(x)$ are the exact solutions of Eqs. (3.12).

The aim of the next algorithm is to implement a procedure to solve (3.11) based on RKHS method that described in Section 3.3.

Algorithm 3.3.9: To approximate the solution of the (3.11) based on RKHS method, there are five main steps:

Input: Integer n; The functions $k_1(x,y)$ and $k_2(x,y)$; The differential operator L; The inner product $\langle u(x), v(x)\rangle_{W_2^3}$.

Output: Approximate solutions $u_n(x)$, $v_n(x)$ of the (3.11).

The steps of the algorithm are

Step A: Fixed x and set $x, y \in [0,1]$;

For $i = 1, 2, ..., n$ do steps $(1, 2\&3)$;

Step 1: Set $x_i = \dfrac{i-1}{n-1}$;

Step 2: If $y \leq x$ then set $K(x,y) = k_1(x,y)$

else set $K(x,y) = k_2(x,y)$;

Step 3 : Set $\psi_i(x) = L_y[K(x,y)]\mid_{y=x_i}$;

Output the orthogonal function system $\psi_i(x)$.

Step B: For $i = 1, 2, ..., n$;

For $j = 1, 2, ..., i$ set $c_{ij} = \langle \psi_j(x), \psi_i(x)\rangle_{W_2^3}$;

Set $\beta_{11} = \dfrac{1}{Sqrt(c_{11})}$;

Output c_{ij} and β_{11}.

Step C : For $i = 2, 3, ..., n$ do steps $(1^\circ \& 2^\circ)$;

Step 1°: For $k = 1, 2, ..., i-1$ set $cc_{ik} = \sum\limits_{m=1}^{k} \beta_{km} c_{im;}$

Step 2°: For $j = 1, 2, ..., i$. If $j \neq i$

$$\text{then set } \beta_{ij} = \frac{-\sum\limits_{k=j}^{i-1} cc_{ik}\beta_{kj}}{\sqrt{c_{ii} - \sum\limits_{k=1}^{i-1} cc_{ik}^2}}$$

$$\text{else set } \beta_{ii} = \frac{1}{\sqrt{c_{ii} - \sum\limits_{k=1}^{i-1} cc_{ik}^2}},$$

Output the orthogonalization coefficients β_{ij}.

Step D: For $i = 1, 2, ..., n$ set $\overline{\psi_i}(x) = \sum\limits_{k=1}^{i} \beta_{ik} \psi_i(x)$;

Output the orthonormal function system $\overline{\psi_i}(x)$.

Step E: Set $u_0(x_1) = 0$;

Set $v_0(x_1) = 0$;

For $i = 1, 2, ..., n$ do steps $(1^*, 2^*, 3^*, 4^*, 5^* \& 6^*)$;

Step 1*: Set $u\left(x_i\right) = u_{i-1}\left(x_i\right)$;

Step 2*: Set $v\left(x_i\right) = v_{i-1}\left(x_i\right)$;

Step 3*: Set $A_i = \sum\limits_{k=1}^{i} \beta_{ik} f(x_k, u_{k-1}\left(x_k\right), u'_{k-1}(x_k), v_{k-1}(x_k), v'_{k-1}(x_k))$;

Step 4*: Set $B_i = \sum\limits_{k=1}^{i} \beta_{ik} g(x_k, u_{k-1}\left(x_k\right), u'_{k-1}(x_k), v_{k-1}(x_k), v'_{k-1}(x_k))$;

Step 5*: Set $u_i\left(x\right) = \sum\limits_{k=1}^{i} A_k \overline{\psi_k}(x)$;

Step 6*: Set $v_i\left(x\right) = \sum\limits_{k=1}^{i} B_k \overline{\psi_k}(x)$.

The n-term approximate solutions $u_n(x)$, $v_n(x)$ of (3.11) is obtained.

Chapter Four
Numerical Results

In this chapter, we use the last algorithm to solve a fuzzy two-point boundary value problems in the reproducing kernel Hilbert space $W_2^3[0,1]$. The algorithm is efficiently applied to solve some models problems with the comparison between the numerical solutions and the exact solutions. It is demonstrated by the numerical examples that this algorithm is of high precision.

The aim of a reproducing kernel Hilbert space method is motivated by the needs for a new numerical method for the solution of the fuzzy boundary value problems with the following characteristics: Firstly, its ability to solve fuzzy boundary value problems without the use of other numerical techniques. Second, it should be of versatile nature. Third, the algorithm should be simple to understand and improvement [44].

Problem 4.1 (**Khastan**) Let us consider the following fuzzy two-point boundary value problem:

$$y''(t) = 2\gamma,\ y(0) = \frac{1}{8}\gamma,\ y(1) = \frac{3}{8}\gamma,\ t \in [0,1], \tag{4.1}$$

where γ is the triangular fuzzy number having $\alpha-$level sets $[\alpha-1, 1-\alpha]$, $\alpha \in [0,1]$.

If y is a $(1,1)-$solution for the problem (4.1), then

$$\left.\begin{array}{l} \underline{y}''_\alpha(t) = 2(\alpha-1),\ \underline{y}_\alpha(0) = \dfrac{\alpha-1}{8},\ \underline{y}_\alpha(1) = \dfrac{3(\alpha-1)}{8}, \\ \overline{y}''_\alpha(t) = 2(1-\alpha),\ \overline{y}_\alpha(0) = \dfrac{1-\alpha}{8},\ \overline{y}_\alpha(1) = \dfrac{3(1-\alpha)}{8}. \end{array}\right\} \tag{4.2}$$

The exact solution:

$$\left.\begin{array}{l} \underline{y}_\alpha(t) = \dfrac{\alpha-1}{8}(8t^2-6t+1), \\ \overline{y}_\alpha(t) = \dfrac{1-\alpha}{8}(8t^2-6t+1). \end{array}\right\} \tag{4.3}$$

Since $\underline{y}_\alpha(t) \leq \overline{y}_\alpha(t)$ for $t \leq \frac{1}{4}$ and $t \geq \frac{1}{2}$, $\underline{y}'_\alpha(t) \leq \overline{y}'_\alpha(t)$ for $t \geq \frac{3}{8}$, $\underline{y}''_\alpha(t) = 2(\alpha-1) \leq$

$2(1-\alpha) = \overline{y}''_\alpha(t)$, and y, $D_1^1 y(t)$, $D_{1,1}^2 y(t)$ have valid level sets for $t \in (\frac{1}{2}, 1)$, then y is $(1,1)-$differentiable on $(\frac{1}{2}, 1)$ and defines a $(1,1)-$solution of the fuzzy differential equation on $(\frac{1}{2}, 1)$.

Using the RKHS method, taking $t_i = \dfrac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 101$, using the reproducing kernel function

$$K_t(y) = \begin{cases} \dfrac{y(1-t)}{18720}(156y^4 + (6t^2 - 4t^3 + t^4)(120 + 30y + 10y^2 - 5y^3 + y^4) \\ \qquad + 12t(360 - 300y - 100y^2 - 15y^3 + 3y^4)), \quad y \le t \\ \dfrac{t(1-y)}{18720}((30ty + 10t^2 y)(-120 + 6y - 4y^2 + y^3) + (120y - 5t^3 y) \\ (36 + 6y - 4y^2 + y^3) + t^4(156 + 36y + 6y^2 - 4y^3 + y^4), \quad y > t \end{cases}$$

on $[0, 1]$, the numerical results of system (4.2) are given in tables 4.1 and 4.2.

Table 4.1: Numerical results for System 4.2 at $t = 0.8$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.165	-0.164999733	2.66665×10^{-7}	-1.61615×10^{-6}
0.1	-0.1485	-0.148499760	2.39999×10^{-7}	-1.61615×10^{-6}
0.2	-0.132	-0.131999786	2.13332×10^{-7}	-1.61615×10^{-6}
0.3	-0.1155	-0.115499813	1.86666×10^{-7}	-1.61615×10^{-6}
0.4	-0.099	-0.098999840	1.59999×10^{-7}	-1.61615×10^{-6}
0.5	-0.0825	-0.082499999	1.33332×10^{-7}	-1.61615×10^{-6}
0.6	-0.066	-0.065999893	1.06666×10^{-7}	-1.61615×10^{-6}
0.7	-0.0495	-0.049499920	7.99998×10^{-8}	-1.61615×10^{-6}
0.8	-0.033	-0.032999946	5.33331×10^{-8}	-1.61615×10^{-6}
0.9	-0.0165	-0.016499973	2.66666×10^{-8}	-1.61615×10^{-6}
1	0	0	0	Indeterminate

Table 4.2: Numerical results for System 4.2 at $t = 0.8$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.165	0.164999733	2.66665×10^{-7}	1.61615×10^{-6}
0.1	0.1485	0.148499760	2.39999×10^{-7}	1.61615×10^{-6}
0.2	0.132	0.131999786	2.13332×10^{-7}	1.61615×10^{-6}
0.3	0.1155	0.115499813	1.86666×10^{-7}	1.61615×10^{-6}
0.4	0.099	0.098999840	1.59999×10^{-7}	1.61615×10^{-6}
0.5	0.0825	0.082499999	1.33332×10^{-7}	1.61615×10^{-6}
0.6	0.066	0.065999893	1.06666×10^{-7}	1.61615×10^{-6}
0.7	0.0495	0.049499920	7.99998×10^{-8}	1.61615×10^{-6}
0.8	0.033	0.032999946	5.33331×10^{-8}	1.61615×10^{-6}
0.9	0.0165	0.016499973	2.66666×10^{-8}	1.61615×10^{-6}
1	0	0	0	Indeterminate

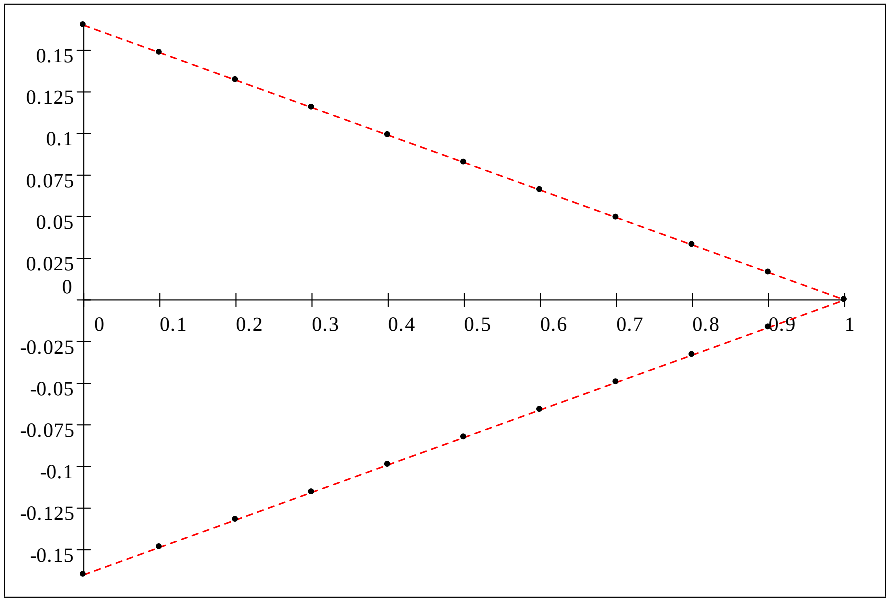

Fig. 1. exact solution (red) and numerical solution (black)

If y is a $(1,2)-$solution for the problem (4.1), then

$$\left.\begin{aligned} \underline{y}''_\alpha(t) = 2(1-\alpha),\ \underline{y}_\alpha(0) = \frac{\alpha-1}{8},\ \underline{y}_\alpha(1) = \frac{3(\alpha-1)}{8}, \\ \overline{y}''_\alpha(t) = 2(\alpha-1),\ \overline{y}_\alpha(0) = \frac{1-\alpha}{8},\ \overline{y}_\alpha(1) = \frac{3(1-\alpha)}{8}. \end{aligned}\right\} \tag{4.4}$$

The exact solution:

$$\left.\begin{aligned} \underline{y}_\alpha(t) &= \frac{-(\alpha-1)}{8}(8t^2-10t-1), \\ \overline{y}_\alpha(t) &= \frac{-(1-\alpha)}{8}(8t^2-10t-1). \end{aligned}\right\} \tag{4.5}$$

Since $\underline{y}_\alpha(t) \le \overline{y}_\alpha(t)$ for $t \in [0,1]$, $\underline{y}'_\alpha(t) \le \overline{y}'_\alpha(t)$ for $t \le \frac{5}{8}$, $\underline{y}''_\alpha(t) \le \overline{y}''_\alpha(t)$ for $t \in (0,1)$, and y, $D_1^1 y(t)$, $D_{1,2}^2 y(t)$ have valid level sets for $t \in (0, \frac{5}{8})$, then y is $(1,2)-$differentiable on $(0, \frac{5}{8})$ and defines a $(1,2)-$solution of the fuzzy differential equation on $(0, \frac{5}{8})$.

Using the RKHS method, taking $t_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 101$, using the reproducing kernel function $K_t(y)$ on $[0,1]$, the numerical results of system (4.4) are given in tables 4.3 and 4.4.

Table 4.3: Numerical results for System 4.4 at $t = 0.1$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.24	-0.239999762	2.37499×10^{-7}	-9.89580×10^{-7}
0.1	-0.216	-0.215999786	2.13749×10^{-7}	-9.89580×10^{-7}
0.2	-0.192	-0.191999810	1.89999×10^{-7}	-9.89580×10^{-7}
0.3	-0.168	-0.167999833	1.66249×10^{-7}	-9.89580×10^{-7}
0.4	-0.144	-0.143999857	1.42499×10^{-7}	-9.89580×10^{-7}
0.5	-0.12	-0.119999881	1.18749×10^{-7}	-9.89580×10^{-7}
0.6	-0.096	-0.095999905	9.49997×10^{-8}	-9.89580×10^{-7}
0.7	-0.072	-0.071999928	7.12498×10^{-8}	-9.89580×10^{-7}
0.8	-0.048	-0.047999952	4.74998×10^{-8}	-9.89580×10^{-7}
0.9	-0.024	-0.023999976	2.37499×10^{-8}	-9.89580×10^{-7}
1	0	0	0	Indeterminate

Table 4.4: Numerical results for System 4.4 at $t = 0.1$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.24	0.239999762	2.37499×10^{-7}	9.89580×10^{-7}
0.1	0.216	0.215999786	2.13749×10^{-7}	9.89580×10^{-7}
0.2	0.192	0.191999810	1.89999×10^{-7}	9.89580×10^{-7}
0.3	0.168	0.167999833	1.66249×10^{-7}	9.89580×10^{-7}
0.4	0.144	0.143999857	1.42499×10^{-7}	9.89580×10^{-7}
0.5	0.12	0.119999881	1.18749×10^{-7}	9.89580×10^{-7}
0.6	0.096	0.095999905	9.49997×10^{-8}	9.89580×10^{-7}
0.7	0.072	0.071999928	7.12498×10^{-8}	9.89580×10^{-7}
0.8	0.048	0.047999952	4.74998×10^{-8}	9.89580×10^{-7}
0.9	0.024	0.023999976	2.37499×10^{-8}	9.89580×10^{-7}
1	0	0	0	Indeterminate

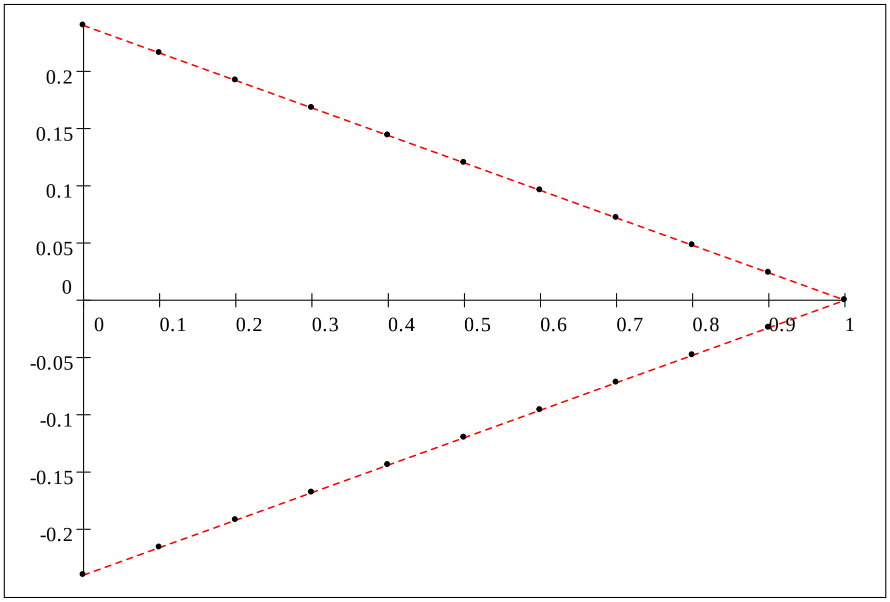

Fig. 2. exact solution (red) and numerical solution (black)

If y is a $(2,2)-$solution for the problem (4.1), then

$$\left.\begin{aligned} \underline{y}''_\alpha(t) = 2(\alpha-1),\ \underline{y}_\alpha(0) = \frac{\alpha-1}{8},\ \underline{y}_\alpha(1) = \frac{3(\alpha-1)}{8}, \\ \overline{y}''_\alpha(t) = 2(1-\alpha),\ \overline{y}_\alpha(0) = \frac{1-\alpha}{8},\ \overline{y}_\alpha(1) = \frac{3(1-\alpha)}{8}. \end{aligned}\right\} \tag{4.6}$$

The exact solution:

$$\left.\begin{aligned} \underline{y}_\alpha(t) = \frac{\alpha-1}{8}(8t^2-6t+1), \\ \overline{y}_\alpha(t) = \frac{1-\alpha}{8}(8t^2-6t+1). \end{aligned}\right\} \tag{4.7}$$

Since $\underline{y}_\alpha(t) \leq \overline{y}_\alpha(t)$ for $t \leq \frac{1}{4}$ and $t \geq \frac{1}{2}$, $\underline{y}'_\alpha(t) \leq \overline{y}'_\alpha(t)$ for $t \leq \frac{3}{8}$, $\underline{y}''_\alpha(t) \leq \overline{y}''_\alpha(t)$ for $t \in (0,1)$, and y, $D^1_2 y(t)$, $D^2_{2,2} y(t)$ have valid level sets for $t \in (0, \frac{1}{4})$, then y is $(2,2)-$differentiable on $(0, \frac{1}{4})$ and defines a $(2,2)-$solution of the fuzzy differential equation on $(0, \frac{1}{4})$.

Using the RKHS method, taking $t_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 101$, using the reproducing kernel function $K_t(y)$ on $[0,1]$, the numerical results of system (4.6) are given in tables 4.5 and 4.6.

Table 4.5: Numerical results for System 4.6 at $t = 0.2$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.015	-0.014999600	3.99998×10^{-7}	-2.66665×10^{-5}
0.1	-0.0135	-0.013499640	3.59999×10^{-7}	-2.66665×10^{-5}
0.2	-0.012	-0.011999680	3.19999×10^{-7}	-2.66665×10^{-5}
0.3	-0.0105	-0.010499720	2.79999×10^{-7}	-2.66665×10^{-5}
0.4	-0.009	-0.008999760	2.39999×10^{-7}	-2.66665×10^{-5}
0.5	-0.0075	-0.007499800	1.99999×10^{-7}	-2.66665×10^{-5}
0.6	-0.006	-0.005999840	1.59999×10^{-7}	-2.66665×10^{-5}
0.7	-0.0045	-0.004499880	1.19999×10^{-7}	-2.66665×10^{-5}
0.8	-0.003	-0.002999920	7.99997×10^{-8}	-2.66665×10^{-5}
0.9	-0.0015	-0.001499960	3.99998×10^{-8}	-2.66665×10^{-5}
1	0	0	0	Indeterminate

Table 4.6: Numerical results for System 4.6 at $t = 0.2$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.015	0.014999600	3.99998×10^{-7}	2.66665×10^{-5}
0.1	0.0135	0.013499640	3.59999×10^{-7}	2.66665×10^{-5}
0.2	0.012	0.011999680	3.19999×10^{-7}	2.66665×10^{-5}
0.3	0.0105	0.010499720	2.79999×10^{-7}	2.66665×10^{-5}
0.4	0.009	0.008999760	2.39999×10^{-7}	2.66665×10^{-5}
0.5	0.0075	0.007499800	1.99999×10^{-7}	2.66665×10^{-5}
0.6	0.006	0.005999840	1.59999×10^{-7}	2.66665×10^{-5}
0.7	0.0045	0.004499880	1.19999×10^{-7}	2.66665×10^{-5}
0.8	0.003	0.002999920	7.99997×10^{-8}	2.66665×10^{-5}
0.9	0.0015	0.001499960	3.99998×10^{-8}	2.66665×10^{-5}
1	0	0	0	Indeterminate

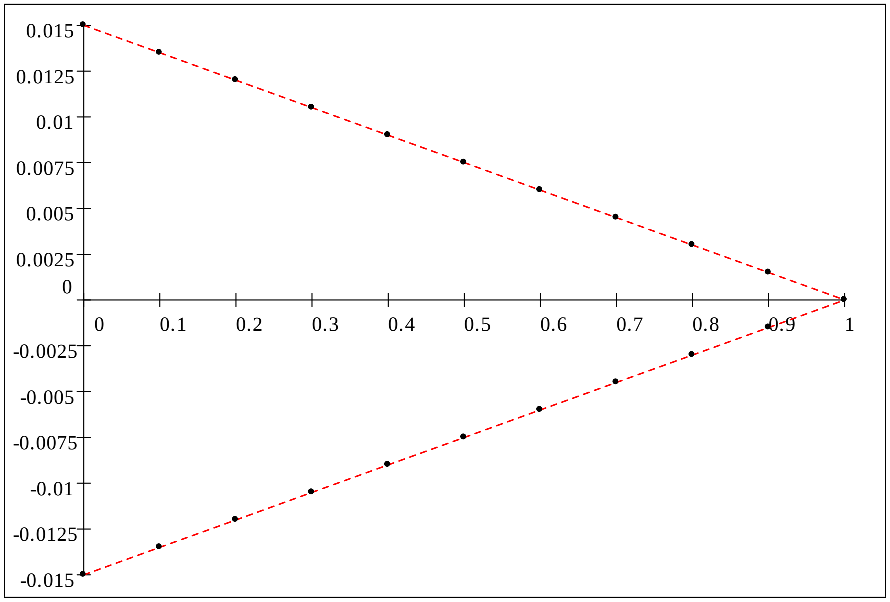

Fig. 3. exact solution (red) and numerical solution (black)

If y is a $(2,1)-$solution for the problem (4.1), then

$$\left.\begin{array}{l} \underline{y}''_\alpha(t) = 2(1-\alpha),\ \underline{y}_\alpha(0) = \dfrac{\alpha-1}{8},\ \underline{y}_\alpha(1) = \dfrac{3(\alpha-1)}{8}, \\ \overline{y}''_\alpha(t) = 2(\alpha-1),\ \overline{y}_\alpha(0) = \dfrac{1-\alpha}{8},\ \overline{y}_\alpha(1) = \dfrac{3(1-\alpha)}{8}. \end{array}\right\} \tag{4.8}$$

The exact solution:

$$\left.\begin{array}{l} \underline{y}_\alpha(t) = \dfrac{-(\alpha-1)}{8}(8t^2-10t-1), \\ \overline{y}_\alpha(t) = \dfrac{-(1-\alpha)}{8}(8t^2-10t-1). \end{array}\right\} \tag{4.9}$$

Since $\underline{y}_\alpha(t) \leq \overline{y}_\alpha(t)$ for $t \in [0,1]$, $\underline{y}'_\alpha(t) \leq \overline{y}'_\alpha(t)$ for $t \geq \frac{5}{8}$, $\underline{y}''_\alpha(t) \leq \overline{y}''_\alpha(t)$ for $t \in (0,1)$, and y, $D_2^1 y(t)$, $D_{2,1}^2 y(t)$ have valid level sets for $t \in (\frac{5}{8}, 1)$, then y is $(2,1)-$differentiable on $(\frac{5}{8}, 1)$ and defines a $(2,1)-$solution of the fuzzy differential equation on $(\frac{5}{8}, 1)$.

Using the RKHS method, taking $t_i = \dfrac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 101$, using the reproducing kernel function $K_t(y)$ on $[0,1]$, the numerical results of system (4.8) are given in tables 4.7 and 4.8.

Table 4.7: Numerical results for System 4.8 at $t = 0.9$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.44	-0.439999862	1.37499×10^{-7}	-3.12499×10^{-7}
0.1	-0.396	-0.395999876	1.23749×10^{-7}	-3.12499×10^{-7}
0.2	-0.352	-0.351999890	1.09999×10^{-7}	-3.12499×10^{-7}
0.3	-0.308	-0.307999903	9.62497×10^{-8}	-3.12499×10^{-7}
0.4	-0.264	-0.263999917	8.24997×10^{-8}	-3.12499×10^{-7}
0.5	-0.22	-0.219999931	6.87498×10^{-8}	-3.12499×10^{-7}
0.6	-0.176	-0.175999945	5.49998×10^{-8}	-3.12499×10^{-7}
0.7	-0.132	-0.131999958	4.12498×10^{-8}	-3.12499×10^{-7}
0.8	-0.088	-0.087999972	2.74999×10^{-8}	-3.12499×10^{-7}
0.9	-0.044	-0.043999986	1.37499×10^{-8}	-3.12499×10^{-7}
1	0	0	0	Indeterminate

Table 4.8: Numerical results for System 4.8 at $t = 0.9$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.44	0.439999862	1.37499×10^{-7}	3.12499×10^{-7}
0.1	0.396	0.395999876	1.23749×10^{-7}	3.12499×10^{-7}
0.2	0.352	0.351999890	1.09999×10^{-7}	3.12499×10^{-7}
0.3	0.308	0.307999903	9.62497×10^{-8}	3.12499×10^{-7}
0.4	0.264	0.263999917	8.24997×10^{-8}	3.12499×10^{-7}
0.5	0.22	0.219999931	6.87498×10^{-8}	3.12499×10^{-7}
0.6	0.176	0.175999945	5.49998×10^{-8}	3.12499×10^{-7}
0.7	0.132	0.131999958	4.12498×10^{-8}	3.12499×10^{-7}
0.8	0.088	0.087999972	2.74999×10^{-8}	3.12499×10^{-7}
0.9	0.044	0.043999986	1.37499×10^{-8}	3.12499×10^{-7}
1	0	0	0	Indeterminate

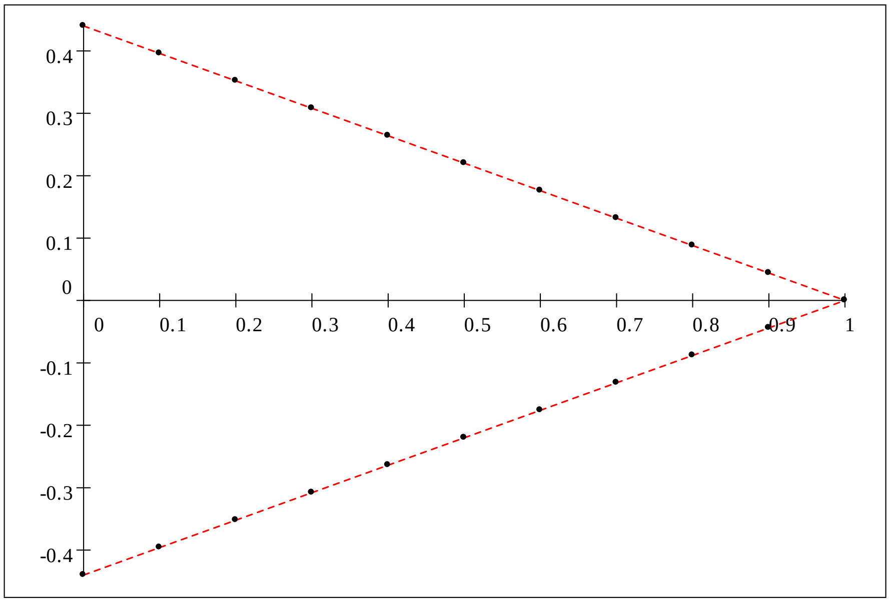

Fig. 4. exact solution (red) and numerical solution (black)

In the next problem, we clear that not necessary four systems of BVPs have solutions.

Problem 4.2 (**Khastan**, **Nieto**) Let us consider the following fuzzy two-point boundary value problem:

$$y''(t) = \gamma_0,\ y(0) = \hat{0},\ y(1) = \gamma_1,\ t \in [0, 1], \tag{4.10}$$

where γ_0 and γ_1 are triangular fuzzy numbers having $\alpha-$level sets $[\alpha, 2-\alpha]$, $[\alpha-1, 1-\alpha]$, $\alpha \in [0, 1]$, respectively.

If y is a $(1, 1)-$solution for the problem (4.2), then

$$\left.\begin{array}{r} \underline{y}''_\alpha(t) = \alpha,\ \underline{y}_\alpha(0) = 0,\ \underline{y}_\alpha(1) = \alpha - 1, \\ \overline{y}''_\alpha(t) = 2 - \alpha,\ \overline{y}_\alpha(0) = 0,\ \overline{y}_\alpha(1) = 1 - \alpha. \end{array}\right\} \tag{4.11}$$

This system has $(1, 1)-$solution:

$$[y(t)]_\alpha = \left[\frac{1}{2}(\alpha t^2 - (2-\alpha)t, \frac{1}{2}((2-\alpha)t^2 - \alpha t)\right], \tag{4.12}$$

on $(0, 1)$.

Using the RKHS method, taking $t_i = \dfrac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 100$, using the reproducing kernel function $K_t(y)$ on $[0, 1]$, the numerical results of system (4.11) are given in tables 4.9 and 4.10.

Table 4.9: Numerical results for System 4.11 at $t = 0.5$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.5	-0.5	0	0
0.1	-0.4625	-0.462499973	2.65710×10^{-8}	-5.74508×10^{-8}
0.2	-0.425	-0.424999946	5.31420×10^{-8}	-1.25040×10^{-7}
0.3	-0.3875	-0.387499920	7.97130×10^{-8}	-2.05711×10^{-7}
0.4	-0.35	-0.349999893	1.06284×10^{-7}	-3.03668×10^{-7}
0.5	-0.3125	-0.312499867	1.32855×10^{-7}	-4.25136×10^{-7}
0.6	-0.275	-0.274999840	1.59426×10^{-7}	-5.79731×10^{-7}
0.7	-0.2375	-0.237499814	1.85997×10^{-7}	-7.83145×10^{-7}
0.8	-0.2	-0.199999787	2.12568×10^{-7}	-1.06284×10^{-6}
0.9	-0.1625	-0.162499760	2.39139×10^{-7}	-1.47162×10^{-6}
1	-0.125	-0.124999734	2.65710×10^{-7}	-2.12568×10^{-6}

Table 4.10: Numerical results for System 4.11 at $t = 0.5$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.25	0.250000531	5.31420×10^{-7}	2.12568×10^{-6}
0.1	0.2125	0.212500504	5.04849×10^{-7}	2.37576×10^{-6}
0.2	0.175	0.175000478	4.78278×10^{-7}	2.73301×10^{-6}
0.3	0.1375	0.137500451	4.51707×10^{-7}	3.28514×10^{-6}
0.4	0.1	0.100000425	4.25136×10^{-7}	4.25136×10^{-6}
0.5	0.0625	0.062500398	3.98565×10^{-7}	6.37704×10^{-6}
0.6	0.025	0.025000371	3.71994×10^{-7}	1.48797×10^{-5}
0.7	-0.0125	-0.012499654	3.45423×10^{-7}	-2.76338×10^{-5}
0.8	-0.05	-0.049999681	3.18852×10^{-7}	-6.37704×10^{-6}
0.9	-0.0875	-0.087499707	2.92281×10^{-7}	-3.34035×10^{-6}
1	-0.125	-0.124999734	2.65710×10^{-7}	-2.12568×10^{-6}

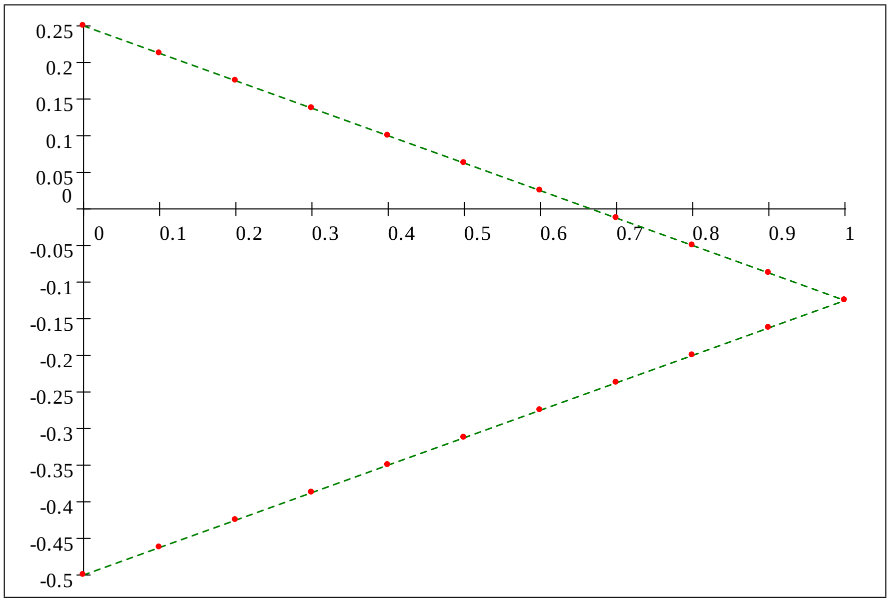

Fig. 5. exact solution (red) and numerical solution (green)

If y is a $(1,2)-$solution for the problem (4.2), then

$$\left.\begin{array}{r}\underline{y}''_\alpha(t) = 2-\alpha,\ \underline{y}_\alpha(0)=0,\ \underline{y}_\alpha(1)=\alpha-1,\\ \overline{y}''_\alpha(t)=\alpha,\ \overline{y}_\alpha(0)=0,\ \overline{y}_\alpha(1)=1-\alpha.\end{array}\right\} \tag{4.13}$$

This system has $(1,2)-$solution:

$$[y(t)]_\alpha = \left[\frac{-1}{2}((\alpha-2)t^2-3\alpha t+4t), \frac{1}{2}(\alpha t^2-3\alpha t+2t)\right], \tag{4.14}$$

on $(0,1)$.

Using the RKHS method, taking $t_i = \dfrac{i-1}{n-1}$, $i = 1,2,...,n$, and $n = 101$, using the reproducing kernel function $K_t(y)$ on $[0,1]$, the numerical results of system (4.13) are given in tables 4.11 and 4.12.

Table 4.11: Numerical results for System 4.13 at $t = 0.4$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.64	-0.639999466	5.33331×10^{-7}	-8.33331×10^{-7}
0.1	-0.588	-0.587999493	5.06665×10^{-7}	-8.61676×10^{-7}
0.2	-0.536	-0.535999520	4.79998×10^{-7}	-8.95519×10^{-7}
0.3	-0.484	-0.483999546	4.53332×10^{-7}	-9.36637×10^{-7}
0.4	-0.432	-0.431999573	4.26665×10^{-7}	-9.87652×10^{-7}
0.5	-0.38	-0.379999600	3.99998×10^{-7}	-1.05263×10^{-6}
0.6	-0.328	-0.327999626	3.73332×10^{-7}	-1.13821×10^{-6}
0.7	-0.276	-0.275999653	3.46665×10^{-7}	-1.25604×10^{-6}
0.8	-0.224	-0.223999680	3.19999×10^{-7}	-1.42857×10^{-6}
0.9	-0.172	-0.171999706	2.93332×10^{-7}	-1.70542×10^{-6}
1	-0.12	-0.119999733	2.66665×10^{-7}	-2.22222×10^{-6}

Table 4.12: Numerical results for System 4.13 at $t = 0.4$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.4	0.4	0	0
0.1	0.348	0.348000026	2.66665×10^{-8}	7.66281×10^{-8}
0.2	0.296	0.296000053	5.33331×10^{-8}	1.80179×10^{-7}
0.3	0.244	0.244000079	7.99997×10^{-8}	3.27868×10^{-7}
0.4	0.192	0.192000106	1.06666×10^{-7}	5.55554×10^{-7}
0.5	0.14	0.140000133	1.33332×10^{-7}	9.52378×10^{-7}
0.6	0.088	0.088000159	1.59999×10^{-7}	1.81818×10^{-6}
0.7	0.036	0.036000186	1.86666×10^{-7}	5.18517×10^{-6}
0.8	-0.016	-0.015999786	2.13332×10^{-7}	-1.33333×10^{-5}
0.9	-0.068	-0.067999760	2.39999×10^{-7}	-3.52940×10^{-6}
1	-0.12	-0.119999733	2.66665×10^{-7}	-2.22222×10^{-6}

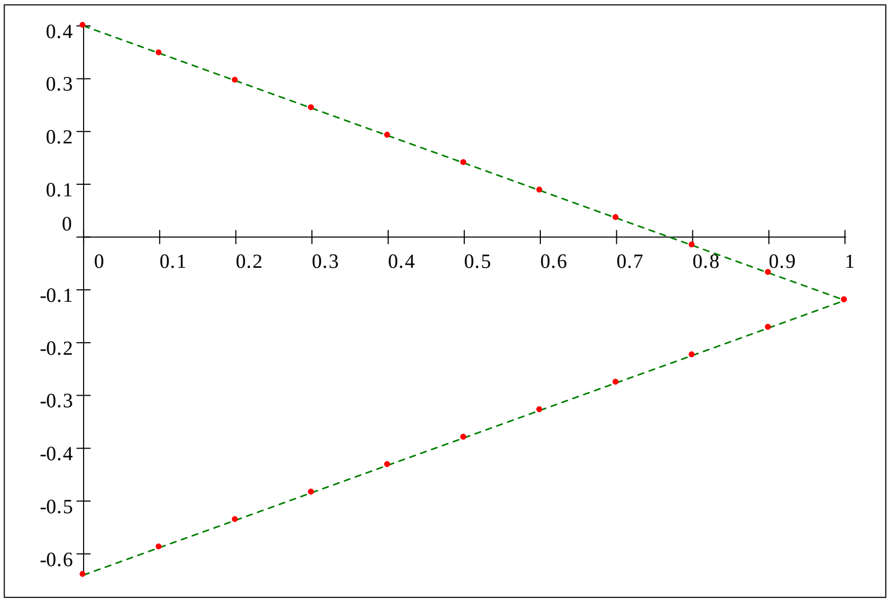

Fig. 6. exact solution (red) and numerical solution (green)

If y is a $(2,2)-$solution for the problem (4.2), then

$$\left.\begin{array}{r}\underline{y}''_\alpha(t)=\alpha,\ \underline{y}_\alpha(0)=0,\ \underline{y}_\alpha(1)=\alpha-1,\\ \overline{y}''_\alpha(t)=2-\alpha,\ \overline{y}_\alpha(0)=0,\ \overline{y}_\alpha(1)=1-\alpha.\end{array}\right\} \tag{4.15}$$

This system has a solution:

$$[y(t)]_\alpha=\left[\frac{1}{2}(\alpha t^2-(2-\alpha)t,\frac{1}{2}((2-\alpha)t^2-\alpha t)\right]. \tag{4.16}$$

Since $\underline{y}_\alpha(t)\geq\overline{y}_\alpha(t)$ for $t\in[0,1]$, then $(2,2)-$solution does not exists.

If y is a $(2,1)-$solution for the problem (4.2), then

$$\left.\begin{array}{r}\underline{y}''_\alpha(t)=2-\alpha,\ \underline{y}_\alpha(0)=0,\ \underline{y}_\alpha(1)=\alpha-1,\\ \overline{y}''_\alpha(t)=\alpha,\ \overline{y}_\alpha(0)=0,\ \overline{y}_\alpha(1)=1-\alpha.\end{array}\right\} \tag{4.17}$$

This system has a solution:

$$[y(t)]_\alpha=\left[\frac{-1}{2}((2-\alpha)t^2-3\alpha t+4t),\frac{1}{2}(\alpha t^2-3\alpha t+2t)\right]. \tag{4.18}$$

Since $\underline{y}'_\alpha(t)\leq\overline{y}'_\alpha(t)$ for $t\leq\dfrac{-1}{2}$, but $t\in(0,1)$ in this case, then $(2,1)-$solution does not exists.

The below problem have two nonlinear systems obtained by fuzzification according to Zadeh's extension. Also, we prove that the RKHS method is good method to solve nonlinear system of the two-point boundary value problems.

Problem 4.3 (**Khastan**, **Nieto**) Let us consider the following fuzzy two-point boundary value problem:

$$y''(t)=y'(t)+\gamma_0,\ y(0)=\hat{0},\ y(1)=\gamma_1,\ t\in[0,1], \tag{4.19}$$

where γ_0 and γ_1 are triangular fuzzy numbers having $\alpha-$level sets $[\alpha,2-\alpha]$, $[\alpha-1,1-\alpha]$, $\alpha\in[0,1]$, respectively.

If y is a $(1,1)-$solution for the problem (4.19), then

$$\left.\begin{aligned} \underline{y}''_\alpha(t) = \underline{y}'_\alpha(t) + \alpha,\ \underline{y}_\alpha(0) = 0,\ \underline{y}_\alpha(1) = \alpha - 1, \\ \overline{y}''_\alpha(t) = \overline{y}'_\alpha(t) + 2 - \alpha,\ \overline{y}_\alpha(0) = 0,\ \overline{y}_\alpha(1) = 1 - \alpha. \end{aligned}\right\} \tag{4.20}$$

The exact solution:

$$\left.\begin{aligned} \underline{y}_\alpha(t) = \frac{1}{e-1}(1 - e^t + \alpha(-2 + 2e^t + t - et)), \\ \overline{y}_\alpha(t) = \frac{1}{e-1}(-3 + 3e^t + 2t - 2et - \alpha(-2 + 2e^t + t - et)). \end{aligned}\right\} \tag{4.21}$$

We see that y, $D_1^1 y(t)$ and $D_{1,1}^2 y(t)$ have valid level sets for $t \in (0,1)$, then y is $(1,1)-$differentiable on $(0,1)$ and defines a $(1,1)-$solution of the fuzzy differential equation on $(0,1)$.

Using the RKHS method, taking $t_i = \dfrac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 251$, using the reproducing kernel function $K_t(y)$ on $[0,1]$, the numerical results of system (4.20) are given in tables 4.13 and 4.14.

Table 4.13: Numerical results for System 4.20 at $t = 0.6$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.47845399210	-0.47845399210	0	0
0.1	-0.44276319368	-0.44276318060	1.30849×10^{-8}	-2.95528×10^{-8}
0.2	-0.40707239526	-0.40707236909	2.61698×10^{-8}	-6.42878×10^{-8}
0.3	-0.37138159684	-0.37138155758	3.92547×10^{-8}	-1.05699×10^{-7}
0.4	-0.33569079842	-0.33569074608	5.23396×10^{-8}	-1.55916×10^{-7}
0.5	-0.3	-0.29999993457	6.54245×10^{-8}	-2.18082×10^{-7}
0.6	-0.26430920157	-0.26430912306	7.85094×10^{-8}	-2.97036×10^{-7}
0.7	-0.22861840315	-0.22861831156	9.15943×10^{-8}	-4.00643×10^{-7}
0.8	-0.19292760473	-0.19292750005	1.04679×10^{-7}	-5.42583×10^{-7}
0.9	-0.15723680631	-0.15723668855	1.17764×10^{-7}	-7.4896×10^{-7}
1	-0.12154600789	-0.12154587704	1.30849×10^{-7}	-1.07654×10^{-6}

Table 4.14: Numerical results for System 4.20 at $t = 0.6$

α	Exact Solution ($\overline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.23536197631	0.23536171462	2.61698×10^{-7}	1.11189×10^{-6}
0.1	0.19967117789	0.19967092928	2.48613×10^{-7}	1.24511×10^{-6}
0.2	0.16398037947	0.16398014394	2.35528×10^{-7}	1.43632×10^{-6}
0.3	0.12828958105	0.12828935861	2.22443×10^{-7}	1.73392×10^{-6}
0.4	0.09259878263	0.09259857327	2.09358×10^{-7}	2.26092×10^{-6}
0.5	0.05690798421	0.05690778793	1.96273×10^{-7}	3.44896×10^{-6}
0.6	0.02121718579	0.02121700260	1.83188×10^{-7}	8.63398×10^{-6}
0.7	-0.01447361262	-0.01447344252	1.70103×10^{-7}	-1.17527×10^{-5}
0.8	-0.05016441105	-0.05016425403	1.57018×10^{-7}	-3.13008×10^{-6}
0.9	-0.08585520947	-0.08585506553	1.43933×10^{-7}	-1.67648×10^{-6}
1	-0.12154600789	-0.12154587704	1.30849×10^{-7}	-1.07654×10^{-6}

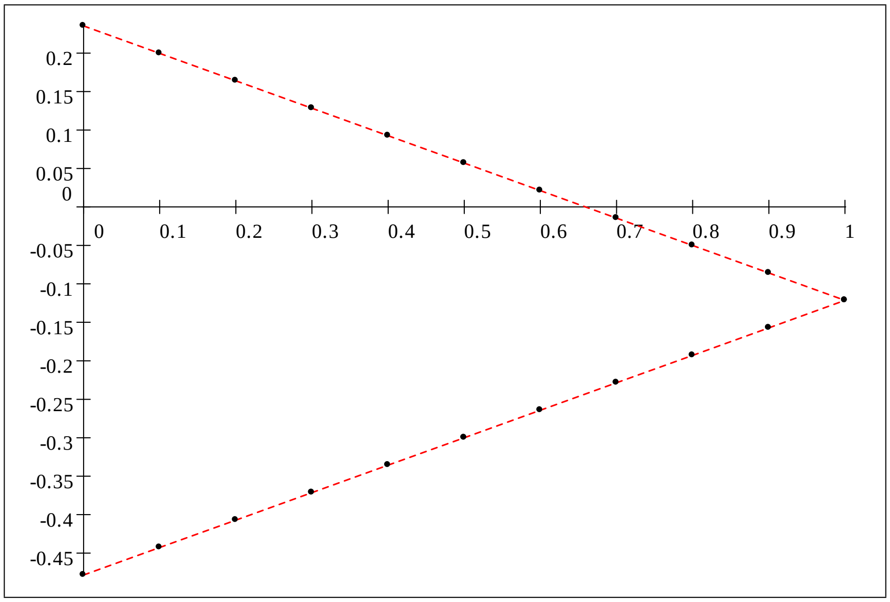

Fig. 7. exact solution (red) and numerical solution (black)

If y is a $(1,2)-$solution for the problem (4.19), then we have a nonlinear system

$$\left.\begin{array}{r} \underline{y}''_\alpha(t) = \overline{y}'_\alpha(t) + 2 - \alpha,\ \underline{y}_\alpha(0) = 0,\ \underline{y}_\alpha(1) = \alpha - 1, \\ \overline{y}''_\alpha(t) = \underline{y}'_\alpha(t) + \alpha,\ \overline{y}_\alpha(0) = 0,\ \overline{y}_\alpha(1) = 1 - \alpha. \end{array}\right\} \tag{4.22}$$

The exact solution

$$\left.\begin{array}{r} \underline{y}_\alpha(t) = \dfrac{1}{e-1}\left[e^{-t}(-2(\alpha-1)e + e^{2t} - e^{1+t}(2+\alpha(t-2)) + e^t(\alpha t - 1))\right], \\ \overline{y}_\alpha(t) = \dfrac{1}{e-1}\left[e^{-t}(2(\alpha-1)e + e^{2t} + e^{1+t}(2+\alpha(t-2)-2t) - e^t(1+(\alpha-2)t))\right]. \end{array}\right\}$$

We see that y, $D_1^1 y(t)$, $D_{1,2}^2 y(t)$ have valid level sets for $t \in (0,1)$, then y is $(1,2)-$differentiable on $(0,1)$ and defines a $(1,2)-$solution of the fuzzy differential equation on $(0,1)$.

Using the RKHS method, taking $t_i = \dfrac{i-1}{n-1}$, $i = 1,2,...,n$, and $n = 251$, using the reproducing kernel function $K_t(y)$ on $[0,1]$, the numerical results of system (4.22) are given in tables 4.15 and 4.16.

Table 4.15: Numerical results for System 4.22 at $t = 0.5$

α	Exact Solution ($\underline{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	-0.8673779936055634	-0.86737835161177135	3.58006×10^{-7}	-4.12745×10^{-7}
0.1	-0.7928861273651927	-0.79288646241517257	3.35049×10^{-7}	-4.22570×10^{-7}
0.2	-0.7183942611248216	-0.7183945732185745	3.12093×10^{-7}	-4.34432×10^{-7}
0.3	-0.643902394884451	-0.6439026840219762	2.89137×10^{-7}	-4.49039×10^{-7}
0.4	-0.56941052864408	-0.5694107948253779	2.66181×10^{-7}	-4.67468×10^{-7}
0.5	-0.49491866240370913	-0.49491890562877972	2.43225×10^{-7}	-4.91445×10^{-7}
0.6	-0.4204267961633382	-0.42042701643218146	2.20268×10^{-7}	-5.23917×10^{-7}
0.7	-0.3459349299229675	-0.34593512723558293	1.97312×10^{-7}	-5.70375×10^{-7}
0.8	-0.2714430636825961	-0.27144323803898498	1.74356×10^{-7}	-6.42331×10^{-7}
0.9	-0.19695119744222547	-0.1969513488423861	1.51400×10^{-7}	-7.68719×10^{-7}
1	-0.12245933120185458	-0.12245945964578811	1.28443×10^{-7}	-1.04887×10^{-6}

Table 4.16: Numerical results for System 4.22 at $t = 0.5$

α	Exact Solution ($\bar{y}$)	Numerical Solution	Absolute Error	Relative Error
0.0	0.6224593312018545	0.62245943232019485	10.1118×10^{-8}	1.62449×10^{-7}
0.1	0.5479674649614836	0.5479675431235962	7.81621×10^{-8}	1.42640×10^{-7}
0.2	0.4734755987211124	0.47347565392699806	5.52058×10^{-8}	1.16597×10^{-7}
0.3	0.398983732480742	0.39898376473039995	3.22496×10^{-8}	8.08295×10^{-8}
0.4	0.3244918662403708	0.324491875533801555	9.29343×10^{-9}	2.86399×10^{-8}
0.5	0.25	0.24999998633720294	1.36627×10^{-8}	5.46512×10^{-8}
0.6	0.17550813375962906	0.17550809714060517	3.66190×10^{-8}	2.08646×10^{-7}
0.7	0.10101626751925848	0.1010162079440067	5.95752×10^{-8}	5.89759×10^{-7}
0.8	0.02652440127888709	0.026524318747408626	8.25314×10^{-8}	3.11153×10^{-6}
0.9	-0.047967464961483665	-0.047967570449190006	1.05487×10^{-7}	-2.19915×10^{-6}
1	-0.12245933120185458	-0.12245945964578811	1.28443×10^{-7}	-1.04887×10^{-6}

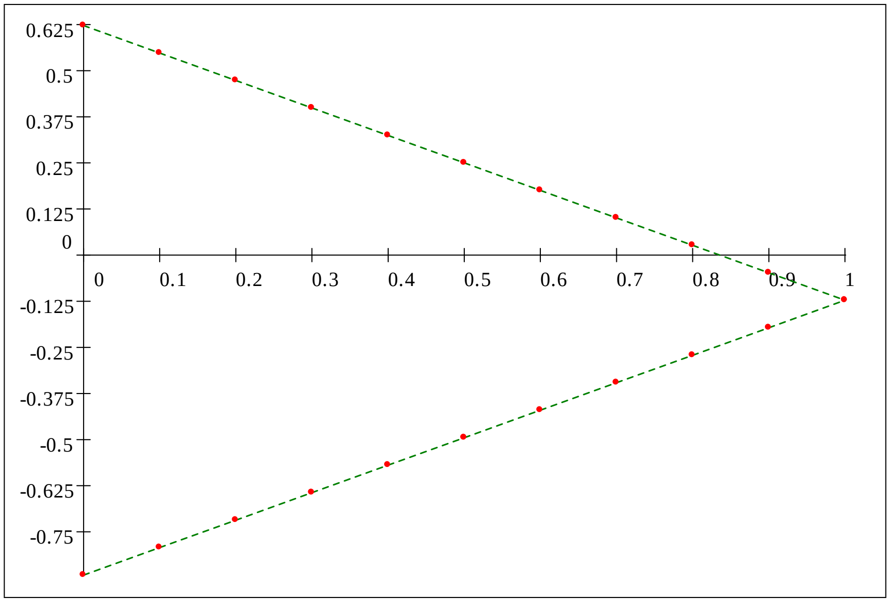

Fig. 8. exact solution (green) and numerical solution (red)

If y is a $(2,1)-$solution for the problem (4.19), then

$$\left.\begin{aligned} \underline{y}''_\alpha(t) &= \underline{y}'_\alpha(t) + 2 - \alpha,\ \underline{y}_\alpha(0) = 0,\ \underline{y}_\alpha(1) = \alpha - 1, \\ \overline{y}''_\alpha(t) &= \overline{y}'_\alpha(t) + \alpha,\ \overline{y}_\alpha(0) = 0,\ \overline{y}_\alpha(1) = 1 - \alpha. \end{aligned}\right\} \tag{4.24}$$

This system has a solution:

$$\left.\begin{aligned} \underline{y}_\alpha(t) &= \frac{1}{e-1}(-1 + e^t + (2-\alpha)t - (2-\alpha)et), \\ \overline{y}_\alpha(t) &= \frac{1}{e-1}(-1 + e^t + \alpha t - \alpha et). \end{aligned}\right\} \tag{4.25}$$

We can see y is not $(2,1)-$differentiable on $(0,1)$. Hence, no $(2,1)-$solution exists for $t \in [0,1]$.

If y is a $(2,2)-$solution for the problem (4.19), then we have a nonlinear system

$$\left.\begin{aligned} \underline{y}''_\alpha(t) &= \overline{y}'_\alpha(t) + \alpha,\ \underline{y}_\alpha(0) = 0,\ \underline{y}_\alpha(1) = \alpha - 1, \\ \overline{y}''_\alpha(t) &= \underline{y}''_\alpha(t) + 2 - \alpha,\ \overline{y}_\alpha(0) = 0,\ \overline{y}_\alpha(1) = 1 - \alpha. \end{aligned}\right\} \tag{4.26}$$

This system has a solution:

$$\left.\begin{aligned} \underline{y}_\alpha(t) &= \frac{1}{e-1}(-1 + e^t + (2-\alpha)t - (2-\alpha)et), \\ \overline{y}_\alpha(t) &= \frac{1}{e-1}(-1 + e^t + \alpha t - \alpha et). \end{aligned}\right\} \tag{4.27}$$

We can see y is not $(2,2)-$differentiable on $(0,1)$. Hence, no $(2,2)-$solution exists for $t \in [0,1]$.

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NUMERICAL SOLUTION OF FUZZY BOUNDARY VALUE PROBLEMS VIA REPRODUCING KERNEL HILBERT SPACE METHOD

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ABSTRACT IN ARABIC

الحل العددي لمسائل القيم الحدّية الضبابية باستخدام طريقة استنساخ نواة فضاء هلبرت

إعداد

غالب ناصر فارس جمعه

المشرف

الأستاذ الدكتور نبيل الشواقفة

ملخص

في هذه الرسالة سنفسر مسائل القيم الحدية الضبابية باستخدام المفهوم العام للاشتقاق الضبابي ، أيضاً سنتحرى مسألة إيجاد حلول عددية تقريبية لمسائل القيم الحدية الضبابية كما سنثبت من خلال الرسالة أن طريقة استنساخ نواة فضاء هيلبرت يمكن تطبيقها لحل مسائل القيم الحدية الضبابية عددياَ.